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THIRD BOOK

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JUNIOR
HIGH SCHOOL MATHEMATICS

THIRD BOOK

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PREFACE

In the teaching of arithmetic we have learned to give content to the symbols before giving skill in the processes with them. We see to it that the child can think six marbles, three marbles, and their sum, nine marbles, before we begin practice with the symbols 6 and 3 to memorize their sum. But we have usually been satisfied to teach algebra as a science of symbols without content. The pupil learns that $a^2 \cdot a^3 = a^5$ and sees that that is according to the rules of the game with these new symbols, but usually he is entirely in the dark as to the meaning of this new symbol, a, and the conditions under which anyone would want to use it. *The distinct aim of the Junior High School Mathematics is to make algebra mean something before giving the technique of its symbols.*

In these books it has been our purpose to extend the pupil's notion of number by giving: (1) The notion of a general number and its symbols; (2) the notion of a variable number; (3) the notion of the relations between variables and the methods of expressing these relations by tables, by graphs, and by formulas. The notion of general number is developed through using numerous concrete instances already in the pupil's experience. He knows that the area of *any* rectangle equals the product of its length and width; that the interest on *any* principal is the product of the principal, the rate, and the time; that 2 times *any number* added to 3 times the *same number* gives 5 times *that number*; and with these and numerous similar experiences fresh in mind he is ready for the new symbols with which to express them. The explanation of number puzzles, and of the peculiarities of the decimal notation, in the first chapter of this book, add an immediate zest to the use of the new symbols. This leads at once to the making and interpreting of formulas, which in our judgment is by far the most important use which the pupil will make of algebra in later life.

The technique of algebra is centered about the transformation of equations. Frequently these equations are formulas, and the transformation has for its purpose the discovery of another relation among the general numbers involved. Often the equation is derived from a familiar formula, and this furnishes a purpose for its solution. It has been our purpose to develop as much skill in the technique of algebraic symbols, — the fundamental processes, factoring, etc., — as is needed to solve the problems that should be given in first-year algebra. To give these processes meaning they are at once used to solve problems. For example, arithmetic progression is taught in connection with addition; geometric progression in connection with multiplication; factoring is connected with multiplication and division. Factoring is used also to put expressions into better form for computation. For instance, the pupil is taught to put the formula, $S = \pi r^2 + \pi rl$, into the more convenient form, $S = \pi r(r + l)$. While unusual emphasis is placed upon a high degree of skill in the types of factoring given, those types not needed in the problem situations of elementary algebra or of life have been omitted, as have also the complicated complex fractions.

The book gives many examples of variation and functions. Relations between variables are expressed in tables of statistics, by graphs, and by formulas. No formal treatment of functions is attempted, but the basis for the general notion is given in many examples, so that the pupil may make use of the idea in his further mathematical study and in his use of mathematics in solving applied problems.

Part I of the Third Book of Junior High School Mathematics relates to algebra and Part II to plane geometry. The algebra contained in this and the two preceding books of this series satisfies the requirements of the National Committee on Mathematical Requirements for first year algebra except that the use of fractional and negative exponents has been omitted. The course in algebra in these books satisfies the requirements for entrance to most colleges. We are confident that this course in algebra, begun very simply in the seventh year and continued through the eighth and the first half of the ninth, gives a much better preparation for further study of mathematics than a single year of algebra in the ninth grade, because of the pupil's longer exposure to algebraic thinking.

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JUNIOR HIGH SCHOOL MATHEMATICS

THIRD BOOK

CHAPTER I

SOME USES OF LETTERS TO REPRESENT NUMBERS

1. How to explain some number puzzles. Many number puzzles are quite mysterious until letters are used to represent the numbers involved. Then the mystery disappears and a simple rule is seen. Try this one. " Take any number you wish; multiply it by 3; add 24; divide this result by 3; subtract the original number. The result is 8. "

It may seem strange that no matter what number you start with you get the same result. You will see why this is true if you call the original number a; multiply it by 3, getting $3a$; add 24, which gives $3a + 24$; divide by 3, which gives $a + 8$; subtract the original number a; the result is 8. This makes it easy to trace the original number a through all the processes and to see how it is eliminated by the last subtraction, leaving 8 as the final result. It can also be seen that the final result depends upon the number added, 24, and upon the number by which it is divided, 3. By changing these numbers the pupil can make up similar puzzles having different answers. For example: Select any number; multiply it by 4; add 36; divide by 4; subtract the original number. The result is 9.

Do you see how such puzzles are made? Try making one.

Exercise 1

1. Explain this puzzle. Take any number, multiply it by 12, add 6, divide by 3, add 14, divide by 4, subtract the original number, and add 9. The result is 13.

2. Think of a number greater than 6. Subtract 6, multiply by 2, add 20, divide by 2, and subtract the first number. The result is 4. Show that this is the correct result.

3. Think the sum of any two numbers; then their difference. Add the sum and the difference. The result is twice the greater number. Subtract the difference of the two numbers from their sum. The result is twice the smaller number. Explain by using letters to represent the numbers.

4. Multiply the sum of any two numbers by their difference. This product is the difference of the squares of the two numbers. Explain, using letters to represent the numbers.

5. Let n be any integer. Is $2n$ an even or an odd number? Answer the same question for $2n + 1$.

6. Let x represent any integer. Using x, write an expression that may represent any even number; also an expression that may represent any odd number.

7. Start with any odd number, add 13, multiply by 2, subtract 8, divide by 4, subtract $4\frac{1}{2}$, and multiply by 2. You have now the number you started with. Show that this is true.

8. Take the number of the month in which you were born. Multiply it by 20, add 12, multiply the result by 5, and add the day of the month on which you were born. If you will now tell me the final result I will tell you your birthday. Make a formula and see if you can explain how I can tell your birthday by this scheme. If you cannot explain it now, see if you can after having worked the next two exercises.

9. How many tens in 47? How many units? How many hundreds in 658? How many tens? How many

units? What is the number that contains 8 hundreds, 5 tens, and 3 units? The number 296 may be written $100 \times 2 + 10 \times 9 + 6$. How may a number be written that contains h hundreds, t tens, and u units? Write a number that contains a hundreds, b tens, and u units.

10. Take a number n; multiply it by 20. Add 7; multiply the result by 5. Add K; subtract 35. You now have $100n + K$. If now $n = 8$, and $K = 22$, $100n + K = 822$. How many hundreds in this number? If the number of hundreds represents the number of the month in which you were born, what month was it? If the remaining part of the number represents the day of the month on which you were born, what day of the month was it? Can you now explain exercise 8?

11. How would you like to have a method of discovering the age of one of your friends? Tell him to take the number of the month in which he was born; multiply by 5; add 13; multiply the result by 20; add his age; add 40; subtract 300. The number represented by the tens' and units' figures is his age, and the other figures give the month in which he was born. Can you make a formula to explain this scheme?

12. Take any three numbers each less than 10. Multiply the first by 2; add 7; multiply the result by 5; add the second number; multiply the result by 10; add the third number; subtract 350. The digits of the final result will be the three numbers you have chosen.

2. Making formulas. The student is likely to make practical use of his knowledge of algebra most frequently in dealing with formulas. He has seen that the formula is usually much shorter and simpler than the rule, that it is easier to apply in solving problems, and easier to check. He has just been making a new use of the formula in solving number puzzles. Other examples of the use of the formula to solve problems will appear as we proceed.

Exercise 2

Write formulas for the following, using the letters indicated:

1. The number of cents, C, in D dollars, d dimes, and c cents.

2. The number N, represented by h hundreds, t tens, and u units.

3. The number of square feet, A, in the plastered surface of a wall l feet long, h feet high, which has one door a feet high and b feet wide, and 2 windows each c feet high and d feet wide.

4. The number of gallons, N, which will accumulate in a standpipe in t minutes if it receives g gallons per minute through one pipe and discharges h gallons per minute through another.

5. The number of pounds, B, of butterfat in p_1 pounds of 4% milk, mixed with p_2 pounds of 5% milk.

6. The number of pounds, B, of butterfat in p_1 pounds of $a\%$ milk, mixed with p_2 pounds of $b\%$ milk.

7. The total number of dollars of discount, D, on two bills of goods, one for a dollars, discounted at $r_1\%$, and the other for b dollars, discounted at $r_2\%$.

8. The total number of dollars, G, gained on two articles, one costing a dollars and sold at a gain of $p\%$, the other costing b dollars and sold at a gain of $q\%$.

9. A man paid rent for a house for 3 years. He paid D dollars a month the first year. The rent was increased i dollars per month the second year, and then h more dollars per month the third year. Write a formula which gives the number of dollars, R, that he paid during the 3 years.

10. Two automobiles start together and go in the same direction, the faster at the rate of r_1 miles per hour and the other at the rate of r_2 miles per hour. Write a formula to give the number of miles, N, that they are apart at the end of t hours.

Exercise 3

te formulas for the following, using the letters in-
d:

Area of a rectangle. (A, b, a.)
Area of a square. (A, s.)
Area of a triangle. (A, b, a.)
Area of a circle. (A, r.)
Circumference of a circle, given the diameter. (c, d.)
Circumference of a circle, given the radius. (c, r.)
Area of a trapezoid. (A, b, b', a).
Perimeter of a regular polygon, given the number of
s and the length of each side. (P, n, s.)
Hypotenuse of a right triangle, given the other two
s. (h, b, a.)
0. One side of a right triangle, given the hypotenuse and
other side. (a, h, b.)
1. Volume of a rectangular parallelopiped. (V, l, w, t.)
2. Volume of a cube. (V, e.)
3. Interest. (i, p, r, t.)
4. Percentage, 3 formulas. (b, r, p.)
5. Distance, rate, and time. (d, r, t.)
6. The volume of a circular cylinder equals π times the
are of the radius times the altitude. (V, r, h.)
7. The volume of a pyramid equals one-third of the prod-
of the base and altitude. (V, b, a.)
8. The surface of a sphere equals four times π times the
are of the radius. (S, r.)
9. The volume of a sphere equals four-thirds of π times
cube of the radius. (V, r.)
0. The altitude of an equilateral triangle equals one-
f of the side times the square root of 3. (a, s.)
1. The area of an equilateral triangle equals one-fourth
he square of the side times the square root of 3. (A, s.)
2. The volume of a cone equals one-third of the prod-
of π, the altitude, and the square of the radius of the
e. (V, h, r.)

3. The decimal system of writing numbers. We write numbers on the scale of 10. Many of the properties of numbers are more easily understood when the numbers are written in a general form. In the following exercises we shall find certain properties of numbers, and also find out the explanations of certain tests of divisibility of numbers written on the scale of 10.

Exercise 4

1. A number is divided by 2 and the quotient is q. Write the number.

2. Write a number which when divided by 3 gives a quotient of d.

3. Write a number which when divided by 8 gives a quotient of x.

4. The divisor is d and the quotient is q. What is the dividend?

5. When we wish to write the general form of a number which is divisible by 2 we often use the form $2n$. Write the general form of a number that is divisible by 3; by 10; by 9; by d; by a.

6. What are the divisors of $6x$? Of $9y$? What is the sum of $6x$ and $9y$? Name a common divisor of $6x$ and $9y$. Is this number a divisor of their sum?

7. Give a common divisor of $100a$ and $10b$. Is it a divisor of their sum?

8. We may use $2a$ and $2b$ as examples of general numbers that are divisible by 2. What is the sum of $2a$ and $2b$? Is this sum divisible by 2?

9. If 2 is a divisor of each of two numbers is it also a divisor of their sum? Illustrate your answer.

10. We have now illustrated this general principle:

A common divisor of two or more numbers is a divisor of their sum.

Illustrate it with other particular numbers.

4. Tests for divisibility by 2, 4, 5, 8 and 10. The pupil has probably used certain tests for divisibility in arithmetic. We shall now show why these tests are true.

The symbols 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 are called *digits*.

Exercise 5

1. Is $10a$ divisible by 2? By 5? By 10? Give the quotient in each case.

2. Which term of $10a + b$ is certainly divisible by 2, no matter what value a or b has? Which term is divisible by 5? By 10?

3. What must be true of b in order that $10a + b$ be divisible by 2? By 5? By 10?

4. We may write $6748 = 1000 \times 6 + 100 \times 7 + 10 \times 4 + 8$. If we write $3569 = 1000T + 100h + 10t + u$, what number does each of the letters T, h, t, and u represent? Tell what number each of these letters represents if 9420 is written in the form $1000T + 100h + 10t + u$. Do the same if 1956 is written in that form; also if 7005 is written in that form; also 32,648; also 326,789.

5. It may be seen from the last exercise that any number may be written in the form $1000T + 100h + 10t + u$, where T stands for the number of thousands, h for the number of hundreds, t for the number of tens, and u for the number of units. What part of $1000T + 100h + 10t + u$ is divisible by 2 no matter what values T, h, t, and u have? Hence what part of the number need be tested to see if the whole number is divisible by 2? Answer the same questions for 5 and 10.

6. From your answers to the questions in the last exercise you should be able to state short methods for testing when a number is divisible by 2, 5, and 10. Do so.

7. Tell which of the following numbers are divisible by 2; by 5; by 10: 876, 345, 9830, 851, 1300.

8. What part of the number $1000T + 100h + 10t + u$

is divisible by 4 no matter what values T, h, t, and u have? Hence what part of the number need be tested to see if the whole number is divisible by 4? Answer the same questions for 8.

9. Now state short methods for testing whether a number is divisible by 4 and 8.

10. Tell which of the following numbers are divisible by 4; by 8: 648, 1372, 7104, 9200, 777,032, 11,262.

5. Tests for divisibility by 3 and 9.

Exercise 6

1. We may write $6534 = 1000 \times 6 + 100 \times 5 + 10 \times 3 + 4 = 999 \times 6 + 99 \times 5 + 9 \times 3 + 6 + 5 + 3 + 4$. The sum of the first three terms, $999 \times 6 + 99 \times 5 + 9 \times 3$, is divisible by 3 and also by 9. Let the pupil find the quotient in each case, without performing any of the indicated multiplications and additions. Find the sum $6 + 5 + 3 + 4$. Is this sum divisible by 3? By 9? From these answers the student should be able to tell whether the whole number is divisible by 3 and by 9. Is it?

2. Use the method of the preceding exercise for testing the number 3281, and determine in this way whether it is divisible by 3 and by 9.

3. We may write any integer of four orders in the form $1000T + 100h + 10t + u = 999T + 99h + 9t + T + h + t + u$. The sum of the first three terms, $999T + 99h + 9t$, is divisible by 3 and by 9. Prove it by performing the divisions. The sum $T + h + t + u$ is the sum of the digits of the number. If the sum of the digits is divisible by 3, what can you say of the number? What can you say if the sum of the digits is divisible by 9?

4. In a similar way any number can be separated into two parts, one of which is divisible by 3 and by 9, the other being the sum of the digits in the number. This gives us short methods for testing whether a number is divisible by 3 or by 9. State these tests.

5. State which of the following numbers are divisible by 3; by 9: 789, 1234, 7560, 7065, 5670, 6431, 111, 7902, 352,801, 2,502,174.

6. A number is *prime* if it has no factors except itself and 1. Find the prime factors of 120, 85, 96, 729, and 424.

7. Find the prime factors of 323.

Hints: Is 323 divisible by 2? By 3? By 5? Do you now know without trial whether 323 is divisible by 4? By 6? By 8? By 9? Why? What number should next be tried as a divisor? What next? See if you can find the prime factors.

8. Find the prime factors of 189, 101, 555, and 728.

9. Find the prime factors of 3000, 10,296, and 2856.

10. Reduce to lowest terms: $\frac{72}{96}$, $\frac{30}{175}$, $\frac{120}{250}$, and $\frac{56}{124}$.

11. Reduce to lowest terms: $\frac{42}{90}$, $\frac{51}{68}$, $\frac{56}{91}$, and $\frac{162}{234}$.

Reduce the following to simplest form. First divide both numerator and denominator by their common factors.

12. $\dfrac{60 \times 21}{7 \times 9}$.

13. $\dfrac{12 \times 15 \times 16}{8 \times 5 \times 9}$.

14. $\dfrac{15 \times 32 \times 88}{120 \times 40 \times 176}$.

15. $\dfrac{156 \times 95 \times 4}{100 \times 133 \times 65}$.

6. Statements of tests for divisibility.

A number is divisible by 2 if the units' digit is divisible by 2.

A number is divisible by 5 if the units' digit is 0 or 5.

A number is divisible by 10 if the units' digit is 0.

A number is divisible by 4 if the number represented by the last two digits is divisible by 4.

A number is divisible by 8 if the number represented by the last three digits is divisible by 8.

A number is divisible by 3 if the sum of its digits is divisible by 3.

A number is divisible by 9 if the sum of its digits is divisible by 9.

Exercise 7

1. The number of births in a town per month is b and the number of deaths is d. Write a formula that gives the increase in the population, i, at the end of m months.

2. The population of this town at a certain time is P'. At this rate of increase what will be the population, P, at the end of m months? Express P in terms of b, d, m, and P'.

3. If a number is divisible by 2 and by 3 is it divisible by 6? Illustrate.

4. Write a general formula for a number which contains units, tens, hundreds, thousands, and ten-thousands.

5. Write two general numbers each of which is divisible by 3; two that are divisible by 11.

6. I know that 2 is a factor of a and that 3 is a factor of b. Is 6 a factor of ab? Is 5 a factor of $a + b$?

7. The remainder when a number is divided by 9 is the same as when the sum of its digits is divided by 9. Show that this is true for 27,842 and for 200,000.

8. Without dividing these numbers by 9 tell what the remainders are when they are divided by 9: 7264, 34,650, 7,296,218, 10,000, 1,010,101.

9. Repeat the tests for divisibility by 2, 4, and 8. Now state a test for divisibility by 16.

10. State a test for divisibility by 25.

11. A number is divisible by a and also by b. Is it divisible by ab?

12. I am thinking of two numbers that are composed of the same digits, but arranged in different orders. The first is divisible by 9. Is the second divisible by 9? Give a reason for your answer.

13. State which of the numbers 2, 3, 4, 5, 6, 8, 9, 10, are factors of any of the following: 648; 1235; 6435; 10,080; 247,132.

CHAPTER II

ADDITION AND SUBTRACTION

7. Addition of positive and negative numbers. The pupil has learned that positive and negative numbers may be represented by distances measured on a line in opposite directions from a fixed point.

Figure 1 gives such a **number scale.**

The fixed point is called the **origin.**

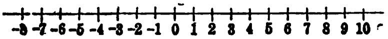

FIG. 1

To add −3 and + 5 on the scale, begin at the origin and count 3 units to the left; from the point reached count 5 units to the right. The distance and direction from the origin to the last point reached give the sum, which is +2.

Exercise 8

Add on the scale:

1. $\begin{array}{r} +7 \\ \underline{-5} \end{array}$

2. $\begin{array}{r} +8 \\ \underline{-8} \end{array}$

3. $\begin{array}{r} -5 \\ \underline{-2} \end{array}$

4. $\begin{array}{r} -8 \\ \underline{+6} \end{array}$

5. $\begin{array}{r} +6 \\ \underline{+3} \end{array}$

6. $\begin{array}{r} -2 \\ \underline{-4} \end{array}$

7. $\begin{array}{r} -6 \\ \underline{+6} \end{array}$

8. $\begin{array}{r} -0 \\ \underline{-6} \end{array}$

9. $\begin{array}{r} -4 \\ \underline{-4} \end{array}$

10. $\begin{array}{r} -2 \\ -3 \\ \underline{+4} \end{array}$

11. $\begin{array}{r} +9 \\ -3 \\ \underline{+1} \end{array}$

12. $\begin{array}{r} +10 \\ \underline{-15} \end{array}$

13. $\begin{array}{r} +10 \\ -8 \\ \underline{-4} \end{array}$

14. $\begin{array}{r} -2 \\ +3 \\ \underline{0} \end{array}$

15. $\begin{array}{r} -0 \\ \underline{+12} \end{array}$

16. $\begin{array}{r} -1 \\ -2 \\ \underline{-5} \end{array}$

17. $\begin{array}{r} -1 \\ +3 \\ \underline{-5} \end{array}$

18. $\begin{array}{r} -7 \\ \underline{+8} \end{array}$

8. Some useful terms. Absolute value. The value of a number without regard to its sign is called its **absolute value.** Thus, the absolute value of $+6$, also of -6, is 6.

The additions made on the number scale illustrate the following definitions of

Sum. The sum of two numbers with like signs is the sum of their absolute values preceded by their common sign.

The sum of two numbers with unlike signs is the difference of their absolute values preceded by the sign of the one having the greater absolute value.

Term. The parts of an algebraic expression to be added are called its **terms.**

Thus in $6a - 4b + 7$, $6a$, $-4b$, and $+7$ are the terms.

Like terms. Terms that have a common factor are called **like terms** with respect to that factor.

Thus, $-4ax$, $12a$, and $7ab^2$ are like terms with respect to the factor a.

Coefficient. If a term is separated into two factors either factor may be called the **coefficient** of the other. Thus, the term abc may be separated into the factors a and bc. Then a is the coefficient of bc and bc is the coefficient of a.

Rule. *To add like terms, multiply their common factor by the sum of its coefficients.*

EXAMPLE. $6x - 4x + bx = (6 - 4 + b)x = (2 + b)x.$

Exercise 9

Add:

1. $5a$ and $3a$.
2. $7x$ and $-5x$.
3. $-4a$ and $-2a$.
4. $15x$ and $-12x$.
5. $10m$, $3m$, and $-9m$.
6. $2ab$, $-3ab$, and $-6ab$.

Collect like terms:

7. $-2a + 3a - 4a.$

8. $7xy - 9xy + xy.$

9. $15n - 10n - 5n.$

10. $12a^2 - 5a^2 - 10a^2.$

11. $18r^2 - 10r^2 + 9r^2.$

12. $60x - 9x - 10x.$

13. $6x - 3 - 4x + 2.$

SOLUTION. $6x - 3 - 4x + 2 = 6x - 4x - 3 + 2$
$= 2x - 1.$

14. $10a - 6b + 3b - 4a.$

15. $15xy - 4ab + 4ab - 3xy.$

16. $12a - 6b - 3 + 5a - b + 6.$

17. $5r^2 + 6r + 7 - r^2 - 9r.$

18. $2 - 5a + 7a^2 - 9 - 10a^2.$

19. $am - cm + 5m.$

SOLUTION. $am - cm + 5m = (a - c + 5)m.$

20. $ax - 3x + dx.$

21. $3r - ar + br.$

22. $r + 2r + r.$

23. $axy - bxy + xy.$

24. $(5x + 6y - 2) + (2x + 9y - 6).$

SOLUTION. This means that $5x + 6y - 2$ and $2x + 9y - 6$ are to be added. Then

$$\begin{aligned}(5x + 6y - 2) + (2x + 9y - 6) &= 5x + 6y - 2 + 2x + 9y - 6 \\ &= 7x + 15y - 8.\end{aligned}$$

25. $2a - b + c + (a - b).$

26. $(6a^2 - 4a - 7) + (6 - 9a - a^2).$

27. $(2ax - 4a + 9) + (2a - 9ax - 2).$

28. $(3mn + 5m - 7) + (8 - 3m - 4mn).$

Add:

29.
$$\begin{array}{r} 5x + 6y - 4 \\ 2x - 9y - 10 \\ x - 2y + 7 \\ \hline \end{array}$$

30.
$$\begin{array}{r} 10a^2 - 5ab + 6b^2 \\ a^2 + 2ab + b^2 \\ 4a^2 - 6ab - 3b^2 \\ \hline \end{array}$$

31.
$$\begin{array}{r} a^3 - 5a^2 + a + 4 \\ 6a^3 - 9a^2 - 4a + 6 \\ -2a^3 + 12a^2 - 2a + 9 \\ \hline \end{array}$$

32.
$$\begin{array}{r} 3.2a^2 - .7a + 8 \\ .34a^2 + 4a - .04 \\ -.05a^2 - 6a + 9.6 \\ \hline \end{array}$$

33. $4xy - 5x^2 + 6y^2,\ 2x^2 - 9xy + y^2,\ 12xy - y^2 + 4x^2.$

9. Subtraction. Subtraction is the process of finding one of two addends when the sum and the other addend are given.

The **minuend** is the given sum.

The **subtrahend** is the given addend.

The **remainder** or **difference** is the addend to be found.

In the subtraction $12 - 8 = 4$, which number is the minuend? Which is the subtrahend? The remainder?

The sum is 8. One addend is 3. What is the other addend? Which number is the minuend? Which is the subtrahend? Which is the remainder?

The sum is 12. One addend is 15. What is the other addend?

The sum is -9. One addend is -6. What is the other addend?

The sum is 6. One addend is -10. What is the other addend?

What must be added to 3 to give 8? To 15 to give 12? To -6 to give -9? To -10 to give 6?

In each of the following exercises subtract the lower number from the upper by finding the number which added to the subtrahend gives the minuend:

$$\begin{array}{ccccccccc} +9 & -16 & +6 & -5 & +3 & -4 & -8 & -7 & 0 \\ \underline{+4} & \underline{-7} & \underline{+8} & \underline{-7} & \underline{-2} & \underline{+2} & \underline{-2} & \underline{-7} & \underline{-6} \end{array}$$

Check the preceding results by adding the subtrahend and the remainder.

In each of the preceding exercises think the sign of the subtrahend changed and add the result to the minuend. Compare these results with those already found. This comparison suggests the following

Rule for subtraction. *Change the sign of the subtrahend and add.*

It follows from this rule that subtracting a number is equivalent to adding a number of the same absolute value but opposite sign.

Exercise 10

Subtract:

1. $\begin{array}{r} +9 \\ \underline{+4} \end{array}$

2. $\begin{array}{r} +2 \\ \underline{+9} \end{array}$

3. $\begin{array}{r} -6 \\ \underline{-9} \end{array}$

4. $\begin{array}{r} -c \\ \underline{+2c} \end{array}$

5. $\begin{array}{r} +6b \\ \underline{+9b} \end{array}$

6. $\begin{array}{r} +8x \\ \underline{-9x} \end{array}$

7. $\begin{array}{r} -5m^2 \\ \underline{-12m^2} \end{array}$

8. $\begin{array}{r} -12ab \\ \underline{-9ab} \end{array}$

9. $\begin{array}{r} -9x \\ \underline{-9x} \end{array}$

10. $\begin{array}{r} -15xy \\ \underline{+15xy} \end{array}$

11. $\begin{array}{r} +7ab \\ \underline{-7ab} \end{array}$

12. $\begin{array}{r} -1 \\ \underline{-0.1} \end{array}$

13. $\begin{array}{r} 0 \\ \underline{-6} \end{array}$

14. $\begin{array}{r} -0.4 \\ \underline{-0.06} \end{array}$

15. $\begin{array}{r} +0.16 \\ \underline{-3} \end{array}$

16. $\begin{array}{r} 3a - 3b - 6c \\ \underline{a - 4b + 9c} \end{array}$

17. $\begin{array}{r} -9c^3 + 5c^2 - 8c \\ \underline{9c^3 - 4c^2 + 6c} \end{array}$

18. $\begin{array}{r} 16x^3y - 4xy^2 - 3y^3 \\ \underline{-5x^3y - 4xy^2 + 6y^3} \end{array}$

19. $\begin{array}{r} a - b - c + d \\ \underline{-a - b + c + d} \end{array}$

20. $\begin{array}{r} \frac{1}{2}x + \frac{3}{4}y - 2\frac{1}{2}z \\ \underline{-\frac{2}{3}x - \frac{1}{2}y + \frac{3}{4}z} \end{array}$

The expression $(12a - b) - (3a + 2b - c)$ means that $3a + 2b - c$ is to be subtracted from $12a - b$. Since to subtract we change signs of the terms of the subtrahend and add, $(12a - b) - (3a + 2b - c) = 12a - b - 3a - 2b + c = 9a - 3b + c$.

21. $-8 - (-5) = ?$

22. $-8 - (+5) = ?$

23. $-3 - (+3) = ?$

24. $-3 - (-3) = ?$

25. $0 - (+2) = ?$

26. $0 - (-2) = ?$

27. $15 - (+6) = ?$

28. $8a - (+3a) = ?$

29. $-6x - (-4x) = ?$

30. $2x - (-4x) = ?$

31. $7 - (-2) - (+4) = ?$

32. $-a - 6a - (-10a) = ?$

33. $2a - 3b - (-5b) = ?$

34. $(8y - 7) - (2y - 3) = ?$

35. $0 - (a - b) = ?$

36. $0 - (2x^2 + 6) = ?$

37. $5x^2 - (3 - x^2) = ?$

38. $y^2 - 2y - (y^2 - y + 3) = ?$

39. $6x^3 - 7x^2 - (x^3 + x^2 - 8) = ?$

40. $3rw + 15w^2 + (42 - 6w^2 + 4r) = ?$

41. $mn - 4 + (mn - 6) = ?$

42. $x^3 - x^2 + x - 8 - (x^2 + x - 1) = ?$

43. $-a + b - c - d + (-b + a - c + d) = ?$

44. $0 - (23 + 16a^2 - 4a^3) = ?$

45. $-5m + y - (-m + y - 9) = ?$

10. Removal of parentheses. The following rules have been illustrated in the preceding problems:

Parentheses preceded by a plus sign may be removed without changing the signs of terms within the parentheses.

EXAMPLE. $a + (b - c + d) = a + b - c + d.$

Parentheses preceded by a minus sign may be removed by changing the sign of each term within the parentheses.

EXAMPLE. $a - (b - c + d) = a - b + c - d.$

Exercise 11

Remove the parentheses and combine like terms in the following:

1. $3 - (-6 - 2).$
2. $7 - (5 - 8 + 2 - 6).$
3. $4 + 6 - 8 - (7 - 12 - 3).$
4. $6 - 2 + (-8 + 3) - (7 - 4).$
5. $2a - 3 - (a + 3).$
6. $y - (3y + 4) + (4y - 3).$
7. $4a - [3a + (b - a)].$
8. $6x - 8 + [7 - 5x - (4x + 6)].$
9. $-(5x + 4y - 8z).$
10. $-9s - [-4r + s - (-5s + s - 9)].$
11. $7a - (a - 3) + (8 - 3a).$
12. $3x - 6 + (5 - x) - (-x + 7).$
13. $-[12 - (2m + 6n) - 5n + m].$
14. $a - b - [(a - b) + (a - b)].$
15. $1 - [2 + (3 - 4) + 5].$
16. $[2a - (b - 2c)] + [-(a + c) + 3b].$

17. By using parentheses indicate that $x^2 - 3x$ is to be subtracted from $5x - 6$.

18. Indicate that $m - 2n + 1$ is to be subtracted from $2m - 3$, and the result subtracted from $2n$. Perform the indicated operations.

19. A rectangular box is x in. long, y in. wide, and z in. deep. Another box is 2 in. longer, 1 in. narrower, and 5 in. deeper. How much more binding would be required for the edges of the second than for the first?

20. By what amount is $x - y$ greater than $a + 3$?

21. What is the larger part of m^2 if $m - n + 1$ is the smaller part?

22. What number is $4t^2 + t + 6$ less than 420? Find the number if $t = 10$?

23. The sides of a rectangle are $x + y - 4$ and $2x - y + 7$. Find the perimeter.

24. How much greater than a^2 is a^3? Answer this question when a has the values 3, -2, 0, and $\frac{1}{3}$. What is the meaning of a negative answer?

25. Find a value of x that will make $8x^3$ equal to, one that will make it less than, and one that will make it greater than, $7x^3$.

26. If a is greater than b, is $a - b$ a positive or a negative number? Is 0 greater or less than -6? How much? Is -12 greater or less than -8? How much?

27. What must be subtracted from $a - \frac{1}{2}b + \frac{2}{3}c$ to leave $\frac{1}{3}a - b + \frac{2}{3}c$?

28. What subtracted from 15 leaves $a - 3b + 7$?

29. If we reverse the order of the terms in the right hand member of

$$s = x + (x + a) + (x + 2a) + (x + 3a),$$

we get

$$s = (x + 3a) + (x + 2a) + (x + a) + x.$$

Adding these two equations term by term we have,

$$\begin{aligned} 2s &= (2x + 3a) + (2x + 3a) + (2x + 3a) + (2x + 3a) \\ &= 4(2x + 3a). \end{aligned}$$

Solve for s. Find the value of s if $x = 1$ and $a = 2$.

30. Solve for s as in the preceding problem if

$$s = a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d).$$

Find the value of s if $a = 1$, and $d = 1$.

31. Add a and $[a + (n - i)d]$.

32. $(a + d) + [a + (n - 2)d] = ?$

33. $(a + 3d) + [a + (n - 4)d] = ?$

34. $(3a - 5d) + [2a + (n + 7)d] = ?$ How many terms in the first parenthesis? How many terms in the brackets? Add the first terms. What is the common factor of the two second terms?

11. Arithmetic progression. A high school boy was making a contract to work during the vacation. The employer said: "I'll give you \$40 the first month, and I'll increase your salary \$5 each month after that. We'll call 4 weeks a month." The boy suggested, "Since you pay me by the week, give me \$10 the first week, and a weekly increase of \$1.25 after that." "It makes no difference to me," said the employer. "You may have it that way if you want it." So they agreed on the boy's suggestion. Did it make any difference? The boy worked twelve weeks. Make an estimate of the answer and then compute it.

Here is another example of the same kind. One boy agrees to work for a year beginning at \$40 a month, and getting an increase of \$2 a month for each month after the first. Another boy agrees to work for a year beginning at the rate of \$40 a month, with an increase of 50¢ a half-month for each half-month after the first. Which receives more money for his year's work?

You might solve this problem as you did the first, by finding the amount earned each month in the first case and each half month in the second case, and then finding the sums. Since you are to have a number of problems of this kind to solve, it will be worth while to find a formula which may be used as a short cut. Furthermore, deriving this formula will be a good test to see if you understand the preceding work of this chapter.

In this problem, each month's wages are found by adding a fixed sum to the wages of the preceding month. If we call the first month's wages a and the monthly increase d, we shall have for the wages for the 12 months, a, $a + d$, $a + 2d$, $a + 3d$, $a + 4d$, $a + 5d$, $a + 6d$, $a + 7d$, $a + 8d$, $a + 9d$, $a + 10d$, and $a + 11d$.

How many times must d be added to a to get the wages for the third month? For the fourth month? For the eighth month? For the nth month? What are the wages for the fourth month? For the twelfth month? For the nth month?

If we call the last of these n terms l, then the answers to these questions show that

$$l = a + (n - 1)d.$$

How much did the boy, who got an increase of \$2 a month, receive the twelfth month? Use the formula. How much did the boy, who got an increase of 50¢ each half-month, receive the twenty-fourth half-month?

Definitions. The set of terms

$$a, a + d, a + 2d, a + 3d, \cdot \cdot \cdot \cdot$$

is called an **arithmetic progression.**

The **first term** is a.

The **common difference** is d.

The **last term** is l.

The **number** of terms is n.

The **sum** of the terms is s. If we call the sum of the first n terms s_n, then

$$s_n = a + (a + d) + (a + 2d) + (a + 3d) + \cdot \cdot \cdot \cdot + [a + (n - 1)d].$$

It is possible to find a simple formula that will give the sum of such a series of terms as this.

EXAMPLE 1. Let us take first a simple case. Let

$$s = 2 + 4 + 6 + 8 + 10.$$

What is a in this case? What is d? What is n? What is l?

If we reverse the order of terms we have

$$s = 10 + 8 + 6 + 4 + 2.$$

If now we add these two equations, we have

$$2s = 12 + 12 + 12 + 12 + 12 = 5 \times 12.$$

Then

$$s = \tfrac{5}{2} \times 12 = 30.$$

Example 2. Find the value of s in a similar way if

$$s = 3 + 7 + 11 + 15 + 19 + 23 + 27.$$

Can you give a general rule for finding s? Let us now find the general formula.

By reversing the terms in the series

$$s_n = a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) \cdot \cdot \cdot + [a + (n - 3)d] + [a + (n - 2)d] + [a + (n - 1)d],$$

we have

$$s_n = [a + (n - 1)d] + [a + (n - 2)d] + [a + (n - 3)d] + \cdot \cdot \cdot \cdot + (a + 2d) + (a + d) + a.$$

By adding these series term by term we get

$$2s_n = [2a + (n - 1)d] + [2a + (n - 1)d] + [2a + (n - 1)d] + \cdot \cdot \cdot \cdot + [2a + (n - 1)d] + [2a + (n - 1)d].$$

In example 1 we found 5 sums of 12 each. In Example 2 you found 7 sums of 30 each. In this general case there are n sums of $2a + (n - 1)d$ each. This gives

$$2s_n = n[2a + (n - 1)d].$$

Then, $$s_n = \frac{n}{2}[2a + (n - 1)d].$$

This may be written, $$s_n = \frac{n}{2}[a + a + (n - 1)d].$$

But since $$a + (n - 1)d = l,$$

then $$s_n = \frac{n}{2}(a + l).$$

This formula may be stated in the rule:

The sum of an arithmetic series is one-half of the number of terms times the sum of the first and last terms.

Tell what each of the letters s, n, a, and l means in this formula.

We can now easily find the answer to the problem at the beginning of section 11. What are a, n, l, and s in the case where the boy received an increase of \$2 a month?

What are they in the case of the boy who received an increase of 50¢ a half-month? Which boy received more for year's work? Are you certain that your answers are correct? Do they seem reasonable?

In problems involving an arithmetic series we often know the first three terms of the series. How can the common difference then be found? How can you test a series of numbers to see whether they form an arithmetic series?

Exercise 12

Find the common difference and the next two terms in each of the first twelve exercises:

1. $2, 4, 6, 8, \cdots$

2. $5, 8, 11, 14, \cdots$

3. $17, 25, 33, \cdots$

4. $a, a + 4, a + 8, a + 12, \cdots$

5. $x, x + y, x + 2y, x + 3y, \cdots$

6. $-12, -7, -2, 3, \cdots$

7. $9, 5, 1, -3, \cdots$

8. $36, 28, 20, \cdots$

9. $1\frac{1}{2}, 3, 4\frac{1}{2}, \cdots$

10. $\frac{7}{8}, 1\frac{3}{8}, 1\frac{7}{8}, \cdots$

11. $-2, -2\frac{2}{3}, -3\frac{1}{3}, \cdots$

12. $x, x - y + 1, x - 2y + 2, x - 3y + 3, \cdots$

13. In the progression 1, 3, 5, $\cdots\cdot$, find the 10th term.

14. In the progression 4, 8, 12, $\cdots\cdot$, find the 8th term.

15. Find the 20th term of the progression $6, 3, 0, -3, \cdots$

16. Find the 10th term in the progression $0, 75, 150, \cdots$

17. In the progression 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, what is a? What is d? What is n? What is l? What is s?

18. Find l and s if $a = 2$, $n = 7$, and $d = 5$.

19. Find l and s if $a = 60$, $n = 12$, and $d = 3$.

20. Find l and s if $a = 42$, $d = 2$, and $n = 8$.

21. Find the sum of the first 5 terms of the progression $7, 4, 1, -2, \cdots$

22. Find the sum of the first 9 terms of the progression 8, 6, 4, 2, · · ·

23. Find the sum of the first 6 terms of $3\frac{1}{2}$, 8, $12\frac{1}{2}$, 17, · · ·

24. Find the sum of the first 8 terms of 0, 16.1, 32.2, 48.3, · · ·

25. A man enters an office at a salary of $1000 a year. His salary is increased $100 a year each year. What is his salary the tenth year? What is the total amount the firm pays him during the 10 years?

26. An automobile is running at the rate of 1200 ft. per minute. Its speed is increased at the rate of 200 ft. per minute. How fast is it running at the end of 10 minutes? How far has it gone at the end of 10 minutes?

27. A body falls 16 ft. the first second, 48 ft. the second second, and so on, gaining 32 ft. each second. How many feet does it fall in the tenth second? How many feet does it fall in 10 seconds?

28. Show that the sum of the first n odd positive integers is n^2.

29. A ball rolling down a plane that is inclined at an angle of 30°, rolls approximately 8 ft. the first second, 24 ft. the second second, and so on, gaining 16 ft. each second. How far does the ball roll in the eighth second? How far does it roll in 8 seconds?

30. Find the sum of the first n even positive integers.

31. If you save 10 cents the first week in January, 15 cents the second, 20 cents the third, and so on, saving 5 cents more each week, how much will you save in a year?

32. How many strokes does a clock that strikes hours make in 24 hours?

33. Find the sum of all integers divisible by 8 between 1 and 500.

34. In a potato race 20 potatoes are placed in a row 1 yd. apart, the first potato being 1 yd. from the basket. How far must a contestant travel in starting from the basket and bringing the potatoes to the basket one at a time?

Exercise 13. Review Questions

1. Give the two numbers whose absolute value is 7. What is their sum? Their difference?

2. How many terms in the expression $3ax^2 - 7(a + b) - 2c^2$? What are they?

3. $(5m^2n - 7mn^2 + 4m) + (3mn^2 - 4n - 2m^2n) - (3m - 4n + 3m^2n) = ?$

4. One field is $(2m + a)$ rods long and $(3m - a)$ rods wide. Another is $(m + 2)$ rods long and $(3m - 5a)$ rods wide. The first contains how many more square rods than the second? Indicate the processes required without attempting to perform them.

5. In the preceding example if $m = 20$ and $a = 5$, how many more square rods in the first field than in the second?

6. Using the formula for the sum of n terms of an arithmetic progression, find the sum of all the even numbers between 1 and 101.

7. What are the first three positive integers that are divisible by 3? Find the 28th positive integer that is divisible by 3.

8. Write a general even number; a general odd number. Show that the sum of an even number and an odd number is always an odd number; also that the sum of two even numbers or of two odd numbers is always an even number.

9. Define absolute value; sum of numbers with like signs; sum of numbers with unlike signs; like terms; coefficient.

10. State the tests for divisibility by 3, 4, 8 and 9.

11. State the rules for removing parentheses.

12. State the formula $s_n = \frac{n}{2}(a + l)$ as a rule.

CHAPTER III

MULTIPLICATION, DIVISION, AND FACTORING

12. Exponents and powers. The product obtained by using a number one or more times as a factor is called a **power** of that number. Thus, 5^3, or $5 \cdot 5 \cdot 5$, or 125, is a power of 5. In 5^3, 5 is called the **base,** and 3 the **exponent** of the power.

Since x^3 means $x \cdot x \cdot x$ and x^5 means $x \cdot x \cdot x \cdot x \cdot x$, then $x^3 \cdot x^5 = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^8$. This illustrates the

Law of exponents for multiplication. *The product of powers with the same base equals that base with an exponent equal to the sum of the exponents of the powers.*

Do you know a short form of statement of this law? Can you make a formula for this law?

The student has learned that

$$\begin{aligned}(+a) \cdot (+b) &= +ab;\\ (+a) \cdot (-b) &= -ab;\\ (-a) \cdot (+b) &= -ab;\\ (-a) \cdot (-b) &= +ab.\end{aligned}$$

These illustrate the

Law of signs for multiplication. *The product of two numbers with like signs is positive, the product of two numbers with unlike signs is negative.*

Exercise 14

Multiply:

1. $2 \cdot (-3)$.	4. $-2 \cdot 3a$.	7. $(-12) \cdot (-2m^2)$.
2. $-6 \cdot 4$.	5. $-3 \cdot (-4a)$.	8. $a^3 \cdot a^2$.
3. $-8 \cdot (-9)$.	6. $(-9) \cdot (-5x)$.	9. $3a^4 \cdot a^6$.

10. $2x \cdot (-6x^3)$.
11. $(-4x) \cdot (-3x^5)$.
12. $x^2 \cdot (-x) \cdot (-3)$.
13. $3 \cdot 0 \cdot (-6) \cdot 2$.
14. $(-1) \cdot (-2) \cdot (3) \cdot$
15. $(-1)^2$.
16. $(-2)^3$.
17. $(+6)^3$.
18. $(2a)^3$.
19. $(3a)^4$.
20. $(-3x)^2$.
21. $(-2) \cdot (-3)^2$.
22. $(-7xy) \cdot 3x$.
23. $3^2 \cdot 3^3 \cdot 3$.
24. $a^4 \cdot a^5 \cdot a$.
25. $7 \cdot 0 \cdot (-2) \cdot$
26. $2xy \cdot 3x^2y^3$.
27. $4ax \cdot (-6x^2)$.
28. $7x^3y^2 \cdot 5xy^4$.
29. $2x \cdot 3y \cdot 4xy$.
30. $4m^2 \cdot 3mn \cdot n^2 \cdot$
31. $(-1)^5$.
32. -1^5.
33. $(-2)^4$.
34. -2^4.
35. $0 \cdot 3 \cdot 4$.
36. $(-2^2) \cdot (-2)^2$.

13. Division of monomials. Division is the process of finding one of the two factors of a product when the product and the other factor are given.

One of the factors of a^5 is a^2. What is the other factor? $a^5 \div a^2 = ?$

One of the factors of $-6x^4$ is $2x$. What is the other factor? $-6x^4 \div 2x = ?$

Make a rule of exponents for division. Make as accurate and concise a statement as you can, just as if you were giving it to the class to be memorized. State this rule as a formula.

Make a rule of signs for division. State it as well as you can.

Exercise 15

The first number is a product and the second is one of its two factors. Find the other factor. Prove the result by multiplication.

1. 16, 2.
2. −10, 10.
3. $3a^2$, a.
4. $-40x$, $8x$.
5. 0, 3.
6. 6, −12.
7. $-4a^2x^2$, $4a^2x^2$.
8. $10 \cdot 7 \cdot 6$, $7 \cdot 6$.
9. 2^5, 2^2.
10. -5^2, -5.

Find what the first number must be multiplied by to get the second.

11. $-m$, $4mn$.
12. $6ab$, $12ab^2$.
13. $-3a$, $6a^2b$.
14. $2x$, $-6x^2$.

15. $\frac{1}{2}, 3$.
16. $\frac{2}{3}, -4$.
17. $\frac{1}{2}, a$.
18. $\frac{3}{4}, b$.
19. $2a, 3a$.
20. $6a, -3a$.

Find the following quotients:

21. $40m^2n^2 \div 5mn$.
22. $4abc \div ab$.
23. $8 \div \frac{1}{3}$.
24. $2a \div \frac{1}{2}a$.
25. $\frac{1}{2}a \div 2a$.
26. $r^2 \div r$.
27. $.5 \div .01$.
28. $.6a \div 2a$.
29. $.45x^2 \div .05x$.
30. $-.01 \div 100$.

14. Multiplication of polynomials by monomials. A polynomial is multiplied by a monomial by multiplying each of its terms by the monomial and adding the products.

EXAMPLE. $2a(a^2 + 2ab + b^2) = 2a^3 + 4a^2b + 2ab^2$.

Exercise 16

Find the following products:

1. $2(3x + 4y - 5)$.
2. $3a(a^2 + 2ab + b^2)$.
3. $-5(6x^2 - 3x - 4)$.
4. $-3(4x - 9b - c)$.
5. $2a^2(5a^2 - 9a + 7)$.
6. $3(400 + 30 + 7)$.
7. $-5(a - b + c - d)$.
8. $-12(-3b^2 - 4x + 7)$.
9. $r^3(r^2 + r - 1)$.
10. $3x^2y^2(9x^2 - 6xy - y^2)$.
11. $-2^2(3^2 - 2^3 + 4)$.
12. $3^3(2 + 3 - 4)$.

15. Division of polynomials by monomials. How can $35 + 40$ be divided by 5 without first adding 35 and 40?

How can $2a + 3a$ be divided by a without first adding $2a$ and $3a$?

How can $3x^3 + 6x^2 + 15xy^2$ be divided by $3x$?

These examples illustrate the

Rule for dividing a polynomial by a monomial. *To divide a polynomial by a monomial divide each term of the polynomial by the monomial and add the quotients.*

Exercise 17

Find the following quotients, and prove the results by multiplication:

1. $(4x + 6) \div 2.$

2. $(a^2 + a) \div a.$

3. $(6a^2 - 3a + 12) \div 3.$

4. $(48x^3 - 8x^2 + 20x) \div 4x.$

5. $(10^3 + 10^2 - 10) \div 10.$

6. $(7 \cdot 10^3 - 2 \cdot 10^2 + 6 \cdot 10) \div 10.$

7. $(8 \cdot 10^2 - 12 \cdot 10 + 24) \div 4.$

8. $(r^2 + r) \div r.$

9. $(a - b + c - 1) \div (-1).$

10. $(.6r^2 - 2.2r - 6) \div .2.$

11. $(\frac{1}{2}a - \frac{2}{3}b) \div \frac{1}{6}.$

12. $(2.5r^3 - 6r^2 - r) \div \frac{1}{6}.$

13. $(3x^3 - 2x^2 - x) \div (-x).$

14. $(60x^3y^2 - 15x^2y^3) \div 5x^2.$

15. $(12a^4b^2 - 9a^2b^4) \div 3a^2b^2.$

Each of the following expressions has a monomial factor. Find it. By division find the other factor.

16. $6a^3 - 9a^2 + 3a.$

Solution. The monomial factor is $3a$. The other factor is $2a^2 - 3a + 1$. Then

$$6a^3 - 9a^2 + 3a = 3a(2a^2 - 3a + 1).$$

17. $50a^3 - 20a.$

18. $2x + 4y - 8z.$

19. $5x + 20x^2 - 15x^3.$

20. $6x^2y - 8xy^2 + 10x^2y^2.$

21. $4a^2b - 10bc + 6ab.$

22. $8x^3y - 6x^2y^2 + 10xy^3.$

23. $12a^2 - 18a + 6.$

24. $14xy - 7x - 7.$

25. $bxy + 4xy - rxy.$

16. Multiplication by a polynomial. The product of two polynomials is found by multiplying each term of one by each term of the other and adding the products.

EXAMPLE. Multiply $2x - 6$ by $4x + 7$.

SOLUTION.

$$\begin{array}{lll} 2x & -\,6 & \\ 4x & +\,7 & \\ \hline 8x^2 & -\,24x & \\ & \,14x & -\,42 \\ \hline 8x^2 & -\,10x & -\,42 \end{array}$$

Exercise 18

Multiply:

1. $a + b$ by $a + b$.
2. $x - y$ by $x - y$.
3. $c - d$ by $c + d$.
4. $2x - y$ by $2x + y$.
5. $x - 3$ by $x + 3$.
6. $2a - b$ by $3a + b$.
7. $3m - 2$ by $m + 2$.
8. $5x + 2$ by $3x - 4$.
9. $a^2 + 4$ by $a + 2$.
10. $m - 6$ by $m + 6$.
11. $x^2 + x + 1$ by $x - 1$.
12. $a^2 - 2a + 4$ by $a + 2$.
13. $m - 3$ by $m + 7$.
14. $r + 8$ by $r - 5$.
15. $a^3 + 5$ by $a^3 - 3$.
16. $a + b$ by $m + n$.
17. $r + s$ by $t + w$.
18. $2a^2 + 3a - 1$ by $a - 2$.
19. $m^2 - mn + n^2$ by $m + n$.
20. $a + b + c$ by $a + b + c$.
21. $a + b - c$ by $a + b + c$.
22. $x^2 + 7$ by $5x^2 - 4$.
23. $ab + cd$ by $ad - bc$.
24. $\pi r^2 + r$ by $\pi r + 2$.
25. $8a - 9b$ by $8a + 9b$.
26. $a^2 + b^2$ by $a^2 - b^2$.
27. $3^2 - 4^2$ by $3^2 + 4^2$.
28. $5 + a^3$ by $a^3 - 5$.
29. $90 + 5$ by $90 - 5$.
30. $x + y + z$ by $x + y + z$.

Find the following squares:

31. $(a + b)^2$.
32. $(m - n)^2$.
33. $(x - y)^2$.
34. $(c + d)^2$.
35. $(r - 4)^2$.
36. $(t + 6)^2$.
37. $(m + 2n)^2$.
38. $(2k + l)^2$.
39. $(a^2 + 1)^2$.
40. $(m^2 + 6)^2$.

17. Division by a polynomial. A study of the process of multiplying one polynomial by another suggests the method of division by a polynomial.

$$\begin{array}{lll} a+5 & & \\ \underline{a-6} & & \\ a^2+5a & & \\ & -6a & -30 \\ \hline a^2 & -a & -30 \end{array} \qquad \begin{array}{lll|l} a^2 & -a & -30 & a+5 \\ \cline{4-4} \underline{a^2+5a} & & & a-6 \\ & -6a & -30 & \\ & \underline{-6a} & \underline{-30} & \end{array}$$

1. In the multiplication of polynomials it has been found convenient to arrange all the expressions according to the descending powers of a leading letter. The same is true in division.

2. Let this arrangement be made. Then the first term of the product is the product of the first term of the multiplicand by the first term of the multiplier. Hence in division the first term of the quotient is found by dividing the first term of the dividend by the first term of the divisor.

3. The dividend is the sum of the products found by multiplying the divisor by each term of the quotient. Therefore we multiply the divisor by the first term of the quotient and subtract the product from the dividend. The remainder must then be the product of the divisor by the remaining terms of the quotient. Dividing the first term of the remainder by the first term of the divisor we get the second term of the quotient. We then multiply the divisor by the second term of the quotient and subtract the product from the remainder in the dividend. In the above example the second remainder is zero. The division is then complete.

Problems in division may be checked by substituting values for the letters. In this example, if $a = 1$, the dividend becomes -30, the divisor 6, and the quotient -5. The division is then checked. Since division by zero is impossible, values of the letters must not be chosen that will reduce the divisor to zero.

Exercise 19

Divide:

1. $a^2 + 5a + 6$ by $a + 3$.
2. $x^2 - 7x + 12$ by $x - 4$.
3. $a^2 + 2ab + b^2$ by $a + b$.
4. $x^2 - 2xy + y^2$ by $x - y$.
5. $m^2 - 8m + 16$ by $m - 4$.
6. $16u^2 - 40u + 25$ by $4u - 5$.
7. $a^2x^2 + 16ax + 64$ by $ax + 8$.
8. $a^2 - b^2$ by $a - b$.
9. $z^2 - 25$ by $z - 5$.
10. $x^4 - y^2$ by $x^2 - y$.
11. $ax + ay + bx + by$ by $a + b$.
12. $15x^2 - 7x - 36$ by $3x + 4$.
13. $2a^2 + 9a - 35$ by $2a - 5$.
14. $8m^2 + 14m - 15$ by $4m - 3$.
15. $5r^2 - 23r - 42$ by $5r + 7$.
16. $10^2 - 4 \cdot 10 + 2^2$ by $10 - 2$.
17. $3^2 + 2 \cdot 3 \cdot 5 + 5^2$ by $3 + 5$.
18. $15^2 - 31^2$ by $15 - 31$.
19. $mx + my + nx + ny$ by $m + n$.
20. $pr - pk + qr - qk$ by $r - k$.

Find these quotients:

21. $(a^3 - b^3) \div (a - b)$.
22. $(x^3 + y^3) \div (x + y)$.
23. $(m^4 - 16) \div (m^2 + 4)$.
24. $(15m^2 + m - 2) \div (3m - 1)$.
25. $(3x^2 + 17x + 20) \div (3x + 5)$.
26. $\dfrac{6m^2 - 4m + 12}{2}$.
27. $\dfrac{15x^2y + 12xy^2}{3xy}$.
28. $\dfrac{3a^3 - \frac{3}{4}a^2 + 2\frac{1}{2}a}{\frac{1}{2}a}$.
29. $\dfrac{3(a + b)}{(a + b)}$.
30. $\dfrac{2(x + y)^2 + 5(x + y)}{x + y}$.
31. $\dfrac{3(x - 1)^2 - 7(x - 1)}{x - 1}$.
32. $\dfrac{4a(a + b)^2 + 6(a + b)}{2(a + b)}$.

18. Important type products. Certain kinds of products occur so frequently that it is worth while learning rules for forming them.

Square of the sum and of the difference of two numbers. Suppose a and b represent any two numbers. Then $a + b$ is their sum. By actual multiplication we find that

$$(a + b)^2 = a^2 + 2ab + b^2.$$

$$\begin{array}{rrr} & a & +\, b \\ & a & +\, b \\ \hline a^2 & +\, ab & \\ & ab & +\, b^2 \\ \hline a^2 & +\, 2ab & +\, b^2 \end{array}$$

This formula may be stated in words as follows:

The square of the sum of two numbers equals the square of the first plus twice their product plus the square of the second.

The pupil should show by actual multiplication that

$$(a - b)^2 = a^2 - 2ab + b^2.$$

State this formula in words.

These rules may be used in squaring binomials.

EXAMPLE. $(2v + 3u)^2 =$ what?

SOLUTION.

$$(2v + 3u)^2 = (2v)^2 + 2 \cdot 2v \cdot 3u + (3u)^2 = 4v^2 + 12vu + 9u^2.$$

Exercise 20

Square the following:

1. $x + y$.
2. $m - n$.
3. $x + 3$.
4. $a - 5$.
5. $r - 16$.
6. $2r + s$.
7. $3m - n$.
8. $5x - 3y$.
9. $7 - 8$.
10. $12 - 5$.
11. $x^2 + y^2$.
12. $x^3 - y^3$.
13. $2xy - 5$.
14. $5x^2 - 6$.
15. $mx + ny$.
16. $q - 3x$.
17. $1 - 2t$.
18. $.5z - 1$.
19. $.03a + .7$.
20. $m^2n^2 - mn$.
21. $2ab - 5$.

The following are the squares of binomials. Find the binomials.

22. $a^2 + 2ab + b^2$.
23. $p^2 - 2pq + q^2$.
24. $z^2 - 6z + 9$.
25. $x^2 + 14x + 49$.
26. $4y^2 - 4y + 1$.
27. $9m^2 + 6m + 1$.
28. $16r^2 + 24r + 9$.
29. $1 - 18z^2 + 81z^4$.
30. $100a^2 + 40a + 4$.
31. $625 + 500 + 100$.
32. $a^2b^2 - 6ab + 9$.
33. $n^2 + n + \frac{1}{4}$.
34. $\frac{1}{9} + \frac{2}{3}a + a^2$.
35. $.01 - .1x + .25x^2$.
36. $25 + 5 + \frac{1}{4}$.
37. $.25a^2 + .2ab + .04b^2$.
38. $4y^2 - 2y + \frac{1}{4}$.
39. $100 + 60 + 9$.
40. $r^6s^2 - 10r^3s + 25$.
41. $\frac{1}{4}x^2 + x + 1$.

42. What must be true of two terms of a trinomial (usually the first and last), if the trinomial is a square? What must be true of the other term?

Some of the following trinomials are squares. Find them and give their two equal factors.

43. $a^2 - 4ab + 4b^2$.
44. $x^2 + xy + y^2$.
45. $m^2 + 2mn - n^2$.
46. $4a^2 + 10a + 25$.
47. $x^2 + 12x + 6$.
48. $4a^2 + 4ab + 4b^2$.
49. $x^2 + x + 1$.
50. $12xy + 4x^2 + 9y^2$.
51. $28 + 4 + 49$.
52. $x^4 - 6x^2 + 9$.
53. $2x^4 - 12x^2 + 18$.
54. $a^2 + b^2 + c^2$.
55. $a^2 - a + 1$.
56. $4x^4y^4 - 12x^2y^2 + 9$.

57. Which of the expressions in exercises 43–56 are not squares? Make a change in each that will make it a square.

19. The product of the sum and the difference of two numbers. By multiplying we find that

$$(a + b)\ (a - b) = a^2 - b^2.$$

From this formula we get the following rule:

The product of the sum and the difference of two numbers equals the difference of the squares of the two numbers.

Exercise 21

Find the following products:

1. $(x + y)\ (x - y)$.
2. $(m - n)\ (m + n)$.
3. $(x + 3)\ (x - 3)$.
4. $(x + 6)\ (x - 6)$.
5. $(6 + y)\ (6 - y)$.
6. $(a^2 - 5)\ (a^2 + 5)$.
7. $(r + 2s)\ (r - 2s)$.
8. $(5x + 6y)\ (5x - 6y)$.
9. $(10 + 1)\ (10 - 1)$.
10. $(20 + 1)\ (20 - 1)$.
11. $(3x^2 + 9)\ (3x^2 - 9)$.
12. $(a + \frac{1}{2})\ (a - \frac{1}{2})$.

Each of the following binomials is the product of the sum and the difference of two numbers. Find the numbers, their sum, and their difference.

13. $r^2 - s^2$.
14. $x^2 - 4$.
15. $9a^2 - 16b^2$.
16. $36x^2 - 1$.
17. $225a^2b^4 - 64c^2$.
18. $36 - 49$.
19. $1 - 49m^4$.
20. $100 - a^6$.

State the quotients in the following indicated divisions:

21. $(t^2 - 25) \div (t - 5)$.
22. $(n^2 - 16m^2) \div (n + 4m)$.
23. $(64 - 49) \div (8 + 7)$.
24. $(x^4 - y^4) \div (x^2 - y^2)$.
25. $(a^2b^4 - 49) \div (ab^2 + 7)$.
26. $(1 - 25m^4) \div (1 - 5m^2)$.
27. $(4 - 1) \div (2 + 1)$.
28. $(49z^2 - 25n^4) \div (7z - 5n^2)$.
29. $(81m^2 - 100) \div (9m + 10)$.
30. $(100 - 25) \div (10 + 5)$.
31. $(10^4 - 10^2) \div (10^2 + 10)$.
32. $(a^2b^4c^6 - 1) \div (ab^2c^3 - 1)$.
33. $(\frac{1}{4}r^2 - \frac{1}{9}b^2) \div (\frac{1}{2}r + \frac{1}{3}b)$.
34. $(\frac{4}{25}b^4 - 1) \div (\frac{2}{5}b^2 - 1$.

20. Product of two binomials with a common term. We may use as the type of two binomials with a common term the binomials $x + a$ and $x + b$. By actual multiplication we find

$$\begin{array}{l}
x + a \\
\underline{x + b} \\
x^2 + ax \\
\underline{bx + ab} \\
x^2 + (a + b)x + ab
\end{array}$$

or

$$(x + a)\ (x + b) = x^2 + (a + b)x + ab.$$

This formula gives the following rule:

The product of two binomials with a common term equals the square of the common term plus the product of the common term by the sum of the other terms, plus the product of the other terms.

EXAMPLE. Find the product $(x + 5)\ (x + 7)$.

SOLUTION. What is the common term? What is its square? What are the other two terms? What is their sum? Their product? Hence show that

$$(x + 5)\ (x + 7) = x^2 + 12x + 35.$$

Exercise 22

Find the following products using the above formula:

1. $(x + 2)\ (x + 3)$.
2. $(a + 4)\ (a + 7)$.
3. $(m + 9)\ (m + 6)$.
4. $(n - 5)\ (n - 8)$.
5. $(x - 4)\ (x - 2)$.
6. $(r + 6)\ (r - 8)$.
7. $(t - 9)\ (t + 3)$.
8. $(k - 3)\ (k - 3)$.
9. $(r + 5)\ (r + 5)$.
10. $(m + n)\ (m - n)$.
11. $(2x + 7)\ (2x + 3)$.

SOLUTION.

$$(2x + 7)\ (2x + 3) = (2x)^2 + (7 + 3)2x + 7 \cdot 3 = 4x^2 + 20x + 21.$$

12. $(5x + 6)\ (5x + 4)$.
13. $(8a - 5)\ (8a - 2)$.
14. $(3s + 7)\ (3s - 3)$.
15. $(7z - 11)\ (7z + 2)$.

The following trinomials are products of two binomials having a common term. Find the binomials.

16. $a^2 + 5a + 6$.

HINTS. What is the square of the common term? What is the common term? What is the sum of the other two terms? What is their product? The student should now be able to show that

$$a^2 + 5a + 6 = (a + 2)\ (a + 3).$$

17. $r^2 + 7r + 12$.
18. $m^2 - 5m + 6$.
19. $t^2 - 5t - 14$.
20. $y^2 + 3y - 18$.
21. $u^2 - 11u + 30$.
22. $k^2 - k - 12$.
23. $b^4 - 9b^2 - 10$.

21. Factoring. Monomial factors in polynomials. If each term of a polynomial contains the same monomial factor, then this monomial factor is a factor of the polynomial, and the problem is to find the other factor. Such factors have already been found in Article 15. As a type form of such expressions we shall take

$$ax + ay - az = a(x + y - z).$$

Exercise 23

Factor the following:

1. $ax + ab$.
2. $2a + 2$.
3. $x^2 - x$.
4. $3c^2 + 6c$.
5. $12y^3 - 6y^2 + 18y$.
6. $5a^2b^2 + 30a^2b^4 - 10a^3b^2$.
7. $10^3 - 10^2 + 10$.
8. $2^4 - 5 \cdot 2^2 + 6$.
9. $18abx + 15bxy - 6bcx$.
10. $120 + 100 - 60$.
11. $20a^2bc - 16ab^2c + 8abc$.
12. $6\pi r^2 - 12\pi r$.
13. $a^2 + ab - a$.
14. $25a^4 + 5a^3$.
15. $80x^2y^2 - 32x^2y + 8xy^2$.
16. $ab - bc - b$.

22. Trinomial squares. We have seen that

$$a^2 + 2ab + b^2 = (a + b)^2$$

and

$$a^2 - 2ab + b^2 = (a - b)^2.$$

In these trinomials two terms are squares and positive and the other term is twice the product of the square roots of the two squares. When these conditions are satisfied the trinomial is the square of a binomial. The factors of the trinomial square may be found by taking the square root of each of the squares and connecting these square roots by the sign of the other term.

EXAMPLE. Find the factors of $x^2 + 8x + 16$.

SOLUTION. Show that the expression is a square. Are there two terms that are squares? Is the other term twice the product of their square roots? What is the sign of the other term? Show then that

$$x^2 + 8x + 16 = (x + 4)\ (x + 4).$$

Exercise 24

Factor the expressions that are not prime:

1. $x^2 + 4x + 4.$
2. $x^2 - 6x + 9.$
3. $x^2 + 2xy + y^2.$
4. $a^2 - 10ab + b^2.$
5. $r^2 - 24rs + 144s^2.$
6. $16x^4 + 24x^2y + 9y^2.$
7. $9a^2 - 42ab + 49b^2.$
8. $t^4 - 2t^2u^2 + u^4.$
9. $m^2 + 2m + 1.$
10. $n^2 + n + \frac{1}{4}.$
11. $30^2 + 2 \cdot 30 \cdot 7 + 7^2.$
12. $80^2 + 2 \cdot 80 \cdot 4 + 4^2.$

In each of the following expressions replace the parentheses by a term that will make the expression a trinomial square, and find the factors of the resulting expression.

13. $a^2 + (\ \ ) + b^2.$
14. $r^2 - (\ \ ) + 16.$
15. $9d^2 + (\ \ ) + e^2.$
16. $4m^2 - (\ \ ) + 9.$
17. $25 + (\ \ ) + 1.$
18. $r^2 + 2r + (\ \ ).$
19. $y^2 - 6y + (\ \ ).$
20. $z^2 + 20z + (\ \ ).$
21. $t^2 - 8t + (\ \ ).$
22. $n^2 + 14n + (\ \ ).$
23. $z^2 + 8zu + (\ \ ).$
24. $a^2 + 2ab + (\ \ ).$

23. Difference of two squares. We have seen that

$$a^2 - b^2 = (a + b)\ (a - b).$$

Exercise 25

Factor:

1. $m^2 - n^2.$
2. $a^2 - 1.$
3. $b^2 - 4.$
4. $1 - r^2.$
5. $9 - 4.$
6. $x^4 - y^2.$
7. $4t^2 - 25.$
8. $16a^2 - 81.$
9. $a^6 - 1.$
10. $y^4 - 36.$
11. $100 - 121m^4.$
12. $r^4 - 49s^2.$

Exercise 26.

Find the prime factors of the following:

1. $x^2 - xy.$
2. $4h^2 + 4h + 1.$
3. $12x - 6y + 24z.$
4. $16 - m^2.$
5. $a^2 - 2a.$
6. $y^2 - 10y + 25.$
7. $4a^2 - 12a + 9.$
8. $4m^4 + 9n^2.$
9. $4n^4 - 9n^2.$
10. $b^4 - 4b^2c + 4c^2.$
11. $36x^2 + 12x^3 + 16x^4.$
12. $4a^3 - a.$
13. $z^3 - z.$
14. $6x^2y^2 - 12x^2y + 24xy^2.$

24. Trinomials of the form $x^2 + (a + b)x + ab$. We have used $x + a$ and $x + b$ as the type of two binomials with a common term, and have found that

$$x^2 + (a + b)x + ab = (x + a)\ (x + b).$$

The pupil should note that the coefficient of x is the algebraic sum of a and b, and that the last term is their product. For example,

$$\begin{aligned}(x + 5)\ (x + 2) &= x^2 + 7x + 10,\\ (x + 5)\ (x - 2) &= x^2 + 3x - 10,\\ (x - 5)\ (x + 2) &= x^2 - 3x - 10,\\ (x - 5)\ (x - 2) &= x^2 - 7x + 10.\end{aligned}$$

If a trinomial comes under this type, it is possible to find two numbers whose sum is the coefficient of x, and whose product is the last term. Such trinomials can usually be factored by inspection.

Exercise 27

Select the trinomials which are of the form $x^2 + (a + b)x + ab$ and find their factors.

1. $x^2 - 9x + 20$.

SOLUTION. We are to find two numbers whose sum is -9 and whose product is 20. These numbers are -4 and -5. Then

$$x^2 - 9x + 20 = (x - 5)\ (x - 4).$$

2. $x^2 + 5x + 6$. 3. $a^2 - 7a + 12$. 4. $n^2 - n - 2$.

SOLUTION. Here we need to find two numbers whose sum is -1 and whose product is -2. These numbers are -2 and $+1$. Hence

$$n^2 - n - 2 = (n + 1)\ (n - 2).$$

5. $m^2 + m - 6$.
6. $r^2 + 6r + 5$.
7. $r^2 - 6r + 5$.
8. $r^2 - 4r - 5$.
9. $r^2 + 4r - 5$.
10. $z^2 - 13z + 12$.
11. $x^2 + 5x + 2$.

SOLUTION. It is impossible to find two numbers whose sum is 5 and whose product is 2, therefore $x^2 + 5x + 2$ cannot be factored.

12. $m^2 - 9m + 8$. 13. $a^2 - a + 3$. 14. $z^2 + 13z - 48$.

25. Summary of factoring. The pupil will find the following suggestions helpful in factoring:

1. Take out all monomial factors, not forgetting factors expressed in Arabic numerals.

2. If the expression is a binomial, see if it is the difference of two squares, such as $a^2 - b^2 = (a + b)\ (a - b)$.

3. If the expression is a trinomial, see if it is

   (*a*) A trinomial square, such as $a^2 + 2ab + b^2 = (a + b)^2$, or $a^2 - 2ab + b^2 = (a - b)^2$.

   (*b*) The product of two binomials with a common term, as $x^2 + (a + b)x + ab = (x + a)\ (x + b)$.

4. Be sure that the factors you have found are *prime* factors.

Exercise 28. Miscellaneous

Find the prime factors of the following:

1. $x^2 + 5x + 6$.
2. $ax - a$.
3. $a^3 - a$.
4. $3m - 12$.
5. $m^2 - 4m + 4$.
6. $y^2 + 9 + 6y$.
7. $1 - z^2$.
8. $10m - 20n + 15$.
9. $6a^2b + 2ab$.
10. $x^2 + 8x + 15$.
11. $x^2 + 15 - 8x$.
12. $2x - 15 + x^2$.
13. $x^2 - 2x - 15$.
14. $9r^2 - 25$.
15. $7a + 3aw$.
16. $p + prt$.
17. $a^2 + 1$.
18. $\frac{1}{2}gt^2 + vt$.
19. $2c - .01c$.
20. $2m^2 - 20mn + 50n^2$.
21. $3v + 6$.
22. $u^2 - v^2t^2$.
23. $abc - ab$.
24. $r^2 - 2r - 35$.
25. $.35hk + 5k^2$.
26. $100 - 25$.
27. $a^2 - ax + 7a$.
28. $R^2 - r^2$.
29. $k^2l - l^3$.
30. $48r^2s - 12rs + 4r$.
31. $27 + 12x + x^2$.
32. $-p^2q^2 + a^2$.
33. $ax + b$.
34. $36b^2 + 60ab + 25a^2$.
35. $ab - a + b$.
36. $6mnp - 15amp + 9bnp$.
37. $150 + 25a$.
38. $x^2 + xy + y^2$.
39. $\frac{1}{2}at^2 - at$.
40. $ar^2 - br^2 + r$.

26. Reducing expressions to more convenient form for computation. The computations in many problems may be shortened by factoring.

EXAMPLE 1. A rectangular room is 26 ft. long and 35 ft. wide and the ceiling is 14 ft. high. Find the area of the walls.

SOLUTION. There are two walls 14 ft. by 26 ft., and two 14 ft. by 35 ft. Hence the number of square feet in the four walls is

$$2 \times 14 \times 26 + 2 \times 14 \times 35.$$

The computation is shortened by taking the factor 2×14 from each term. We then have

$$2 \times 14 \times 26 + 2 \times 14 \times 35 = 2 \times 14(26 + 35) = 28 \times 61 = 1708.$$

Let the pupil make a formula for finding the area of the walls of a room l feet long, w feet wide, and h feet high. Reduce this formula to the most convenient form for computation.

EXAMPLE 2. To find the volume of the frustum of a cone add the sum of the bases to the square root of their product and multiply the result by one-third of the altitude. Express this rule as a formula and reduce the formula to a convenient form for computation.

SOLUTION. Let r and R be the radii of the bases and h the altitude. Then the volume, V, is given by the formula

$$\begin{aligned} V &= \frac{h}{3}(\pi r^2 + \pi R^2 + \sqrt{\pi r^2 \times \pi R^2}) \\ &= \frac{h}{3}(\pi r^2 + \pi R^2 + \pi r R) \\ &= \frac{\pi h}{3}(r^2 + R^2 + rR). \end{aligned}$$

Let the pupil use this last form and find the volume of a frustum in which $r = 6$, $R = 10$, and $h = 12$.

These examples show that formulas may sometimes be put into much more convenient form for computation by factoring. Thus, $S = \pi r(r + a)$ is more convenient than $S = \pi r^2 + \pi r a$.

Exercise 29

Reduce the formulas in the first six exercises to the most convenient forms for computation:

1. $s = \frac{1}{2}gt^2 + vt$, used for finding the distance passed over by a falling body in a given time. Find s if $g = 32$, $t = 6$, $v = 50$.

2. $S = 2\pi r^2 + 2\pi rh$, the total surface of a right circular cylinder.

3. $S = \pi r^2 + \pi rl$, the total surface of a right circular cone. Find S if $r = 1$, $l = 6$, and $\pi = 3\frac{1}{7}$.

4. $M = 12a^2b^2 - 3ab^3 + 6a^3b$.

5. $R = 3\pi mn^2 + 6\pi mnp$.

6. $Z = pqr^2 - pq^2r + p^2qr$.

7. Find the area and perimeter of Figure 2.

8. Find the area and perimeter of Figure 3.

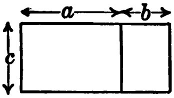

FIG. 2.

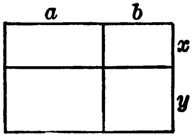

FIG. 3.

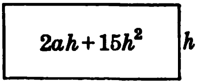

FIG. 4.

9. The altitude of Figure 4 is h and the area is $2ah + 15h^2$. Find the base.

10. Figure 5 is a right triangle with the two legs given. Find the hypotenuse.

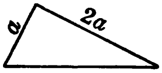

FIG. 5.

11. The area of a square is $x^2 + 6x + 9$. Find the side and the perimeter.

12. Show that the area of Figure 6 is given by the formula $A = (S + s)(S - s)$.

13. Find the area of Figure 6 if $S = 24$ and $s = 9$; if $S = a + b$ and $s = a - b$.

14. Show that the area of the ring in Figure 7 is given by the formula $A = \pi(R + r)(R - r)$.

15. Find the area of the ring in Figure 7 if $R = 35$ and $r = 17$; if $R = a + 5$ and $r = 2a - 6$.

16. Show that the volume of the hollow tile in Figure 8 is given by the formula $V = \pi h(R + r)\ (R - r)$.

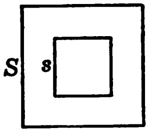

FIG. 6.

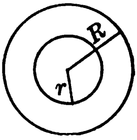

FIG. 7.

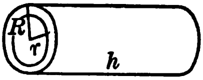

FIG. 8.

17. Find the volume of the hollow tile in Figure 8 if $h = 20$, $R = 16$, and $r = 14$.

18. Figure 9 is a right triangle. Show that

$$b = \sqrt{(h + a)(h - a)}.$$

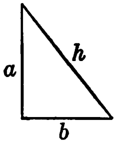

FIG. 9.

27. Solving equations by factoring. Given that $xy = 12$, give three sets of values for x and y that satisfy the equation. If x is $\frac{1}{2}$, what is the corresponding value of y so that $xy = 12$? Give five more sets of values for x and y for which $xy = 12$. How many sets of values are possible?

Give three sets of values for x and y so that $xy = 0$. How many such sets are possible? What is true of each set?

Principle. *The product of two factors is zero when either factor is zero, and only when one of the factors is zero.*

Upon what condition is $(x - 1)\ (x - 2) = 0$?

If $x - 1 = 0$ what is the value of x?

If $x - 2 = 0$ what is the value of x?

We see then that the equation $(x - 1)\ (x - 2) = 0$ is satisfied if $x = 1$ or if $x = 2$.

EXAMPLE. Solve the equation $x^2 - x = 6$.

Transpose all terms to the left member. (1) $x^2 - x - 6 = 0$.
Factor the left member. (2) $(x + 2)\ (x - 3) = 0$.
Then, (3) $x + 2 = 0$,
or (4) $x - 3 = 0$.
From (3), (5) $x = -2$.
From (4), (6) $x = 3$.

Checking each value, $(-2)^2 - (-2) = 6,$
that is $4 + 2 = 6;$
or, $3^2 - 3 = 6,$
that is $9 - 3 = 6.$

Exercise 30

Solve the following equations:

1. $x^2 - 9x + 20 = 0.$
2. $a^2 + 4 = 5a.$
3. $12h = 32 + h^2.$
4. $3n + n^2 = 28.$
5. $s^2 + 5s + 6 = 0.$
6. $y^2 + 7 = 8y.$
7. $r^2 - 4 = -3r.$
8. $x^2 - 5x = 14.$
9. $d^2 = 45 - 4d.$
10. $p^2 = 3p + 28.$
11. $d^2 + 10d + 21 = 0.$
12. $3r - 10 = -r^2.$
13. $s^2 = 9s - 8.$
14. $a^2 + 15a = -54.$
15. $3n^2 = 9n.$
16. $5n^2 - 20n = 0.$
17. $4b^2 - 4b = -1.$
18. $12x^2 - 75 = 0.$
19. $r = r^2 + \frac{1}{4}.$
20. $n(n - 2) = 5(n - 2).$
21. $6(x + 2) + 4 = x^2.$
22. $(S - 3)^2 - 4 = 0.$
23. $(2y + 1)^2 - (y + 3)^2 = 0.$

24. The distance passed over by a body falling from rest is given by the formula $s = 16t^2$, in which t represents the time in seconds and s the distance in feet. If a bullet is dropped from the top of a building 64 ft. high, in how many seconds will it reach the ground?

25. If a bullet were dropped from an airplane 400 ft. high, in how many seconds would it reach the ground?

26. The volume of the hollow tile, Figure 8, page 41, is given by the formula $V = \pi h(R + r)(\mathrm{R} - r)$. Find R if $V = 64\pi$, $h = 4$, and $r = 3$.

27. Also find r if $V = 80\pi$, $R = 7$, and $h = 2$.

28. The formula for the total surface, S, of a right circular cylinder is $S = 2\pi r^2 + 2\pi rh$, in which r is the radius and h the altitude of the cylinder. Find the radius of a cylinder whose altitude is 3 inches and total surface 20π sq. in.

29. Find the radius of a cylinder whose altitude is 7 ft. *and total* surface 88π sq. ft.

28. Geometric progression. A blacksmith was asked his price for shoeing a horse. He answered, "One-half a bushel of wheat for each shoe." When the owner of the horse objected to the price the blacksmith said, "If you think that is too much, pay me one grain of wheat for the first nail, 2 grains for the second, 4 for the third, and so on, doubling the amount for each successive nail." The owner agreed to this. Did he make a good bargain? There are 8 nails in each shoe. A grain of wheat will be counted as weighing 1 grain, and there are 7000 grains in 1 pound avoirdupois. A bushel of wheat weighs 60 pounds.

The blacksmith gets 1 grain for the first nail, 2 grains for the second, 4, or 2^2, for the third, 8, or 2^3, for the fourth, and so on. Write as powers of 2 the number of grains that he will get for each of the first 6 nails; for the tenth; for the last.

It will evidently take considerable multiplying and adding to get the total number of grains of wheat that the blacksmith is to receive. There is a very simple formula which will give the answer to this problem and others like it. We shall now undertake to find this formula.

In this problem we have started with the number 1 and have multiplied it by 2, this result by 2, this last result by 2, and so on, getting each term by multiplying the preceding term by 2.

It will probably be easier if we let a be the first term of such a series, and r the multiplier. Then the terms of the series will be

$$a, ar, ar^2, ar^3, ar^4, \cdots$$

Such a series of terms is called a **geometric progression.**

The **first term** is a.

The **ratio** is r. It may be found by dividing any term by the preceding term.

The **nth term** is ar^{n-1}. If we let l stand for the last of n terms we have $l = ar^{n-1}$.

The **sum** of the first n terms is represented by S_n.

We now wish to find a formula for the sum of the first n terms. Let us first take $n = 8$. Then we have

$$S_8 = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6 + ar^7.$$

If we multiply each member of this equation by r, we have the two equations,

$$S_8 = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6 + ar^7,$$

and

$$rS_8 = \phantom{a +{}} ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6 + ar^7 + ar^8.$$

Subtracting, $S_8 - rS_8 = a - ar^8.$

Factoring, $(1 - r)S_8 = a(1 - r^8).$

Solving for S_8, $S_8 = \dfrac{a(1 - r^8)}{1 - r}.$

Suppose that there are n terms. If we multiply each member of the equation

$$S_n = a + ar + ar^2 + ar^3 + \cdot \cdot \cdot \cdot + ar^{n-2} + ar^{n-1}$$

by r, we have,

$$rS_n = ar + ar^2 + ar^3 + \cdot \cdot \cdot \cdot + ar^{n-1} + ar^n.$$

Subtracting, $S_n - rS_n = a - ar^n.$

Factoring, $(1 - r)S_n = a(1 - r^n).$

Solving for S_n, $S_n = \dfrac{a(1 - r^n)}{1 - r}.$

If we multiply both numerator and denominator of this result by -1, we have

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

Now we may use this formula to find how many grains of wheat it cost to have the horse shod. In that problem, $a = 1$, $n = 32$, and $r = 2$. Then if S denotes the sum,

$$S = 2^{32} - 1.$$

What is the 5th power of 2? The 10th? Then we can find without difficulty that

$$2^{32} = 2^{10}\cdot2^{10}\cdot2^{10}\cdot2^{2} = 1024\cdot1024\cdot1024\cdot4$$
$$= 4{,}279{,}863{,}296.$$

Then the number of bushels of wheat is

$$\frac{4{,}279{,}863{,}295}{7000\cdot60} = 10{,}190 +.$$

Did the blacksmith make a good bargain?

Exercise 31

1. Find the sixth term of the geometric progression 3, 6, 12, · · · · What is the ratio?

2. Find the ratio and the seventh term of 4, 12, 36, · · ·

3. Find the ratio and the tenth term of 4, 2, 1, $\frac{1}{2}$, · · ·

4. Find the tenth term of the series 2430, 810, 270 · ·

5. Find the eighth term of the series $\frac{1}{32}$, $\frac{1}{8}$, $\frac{1}{2}$, · · · ·

6. Find the seventh term of the series $\frac{2}{3}$, 2, 6, · · · ·

7. Find the fourth term of the geometric progression of which $a = 2$, and $r = 4$.

8. Find the fifth term of the geometric progression of which $a = y$, and $r = \frac{1}{2}$.

9. How many consecutive terms of a geometric progression must be given to determine the ratio? How is the ratio then found?

10. If $r = 1.04$, and $a = 200$, find the third term.

11. Indicate the sixth term of the series in the preceding exercise without computing it.

12. The third term of a geometric progression is 50. The first term is 2. Find the ratio.

13. The third term of a geometric progression is 27. The first term is $\frac{3}{4}$. Find the fourth term.

14. If $1 is put at 6% interest, compounded annually, how much is due at the end of 1 year? At the end of 2 years? At the end of 3 years? Show that these answers are in geometric progression. What is r?

15. If a principal, P, is put at interest at 5% for one year, what is due at the end of the year? What must P be multiplied by to find the amount due at the end of the year? If

the amount due at the end of the first year is put at interest at 5% for another year, what is due at the end of this second year? What must P be multiplied by to find this second result?

16. If a principal, P, is put at 4% interest, compounded annually, by what must the amount drawing interest during any year be multiplied to get the amount due at the end of the year?

17. If P dollars are put at interest at 6%, compounded annually, show that the amount due at the end of each of the first 6 years is respectively $(1.06) \cdot P$, $(1.06)^2 \cdot P$, $(1.06)^3 \cdot P$, $(1.06)^4 \cdot P$, $(1.06)^5 \cdot P$, $(1.06)^6 \cdot P$. What is r in this series?

18. Using the formula for the sixth term of a geometric progression, find the amount of $100 at 3% for 5 years compounded annually.

19. A boy deposits $100 in the savings bank. It bears interest at the rate of 4%, compounded semi-annually. How much is due him at the end of 3 years? Show that the amounts due at the ends of successive half-years form a geometric progression in which the ratio is 1.02. What is the first term?

20. A series of five squares is arranged as in Figure 10. The area of each square is one-half of the area of the next larger one. The area of the largest is 36 sq. in. Find the sum of the areas of the five squares.

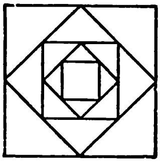

FIG. 10.

21. In reducing $\frac{2}{3}$ to a decimal fraction the result $.66666 \cdots = .6 + .06 + .006 + .0006 + \cdots$. Show that this is a geometric progression by finding the common ratio.

22. Reduce $\frac{5}{11}$ to a decimal fraction of eight places and show that the result is equivalent to the sum of a geometric progression of four terms. What is the ratio?

23. Write the first four terms of the geometric progression whose first term is 1 and common ratio .1. Write the *seventeenth* term of the progression.

Exercise 32. Review Questions

1. Express 2^4 in another way. Show that $2^4 \cdot 3^4 = 6^4$; also that $2^4 \cdot 2^3 = 2^7$.

2. Give the sign of the result in the following: $(-1)^2$; $(-1)^2 \cdot (-1)^3$; $-1 \cdot 1^2 \cdot (-1)$; $-2^4 \cdot (-2)^4 \cdot (-1)^3$.

3. Give two sets of factors of -6; of $+6$. Give as many sets of factors as you can for 12; for 30.

4. Find the sum of 15 terms of the arithmetic progression whose first term is -2 and common difference -1.

5. One factor of $.5x^2 - x$ is $.2x$. Find the other factor.

6. Express the area of a rectangle whose length is the sum of a and b and whose width is the difference of a and b. Find the area of the rectangle.

7. How much larger is a rectangle $(3x + 2)$ feet by $(4x - 1)$ feet than a rectangle $(12x + 1)$ feet by $(x - 2)$ feet? First indicate the processes to be performed, then perform them, using possible short cuts.

8. The radius of the base of a certain cylinder is $(2x - 3)$ inches. Its height is 3 inches greater than the diameter of the base. How much greater is the curved surface of the cylinder than the sum of the areas of its bases? First indicate the processes, then perform them.

9. Of what type is the product $(2x - 3)(2x + 3)$? State the rule for finding this type of product. What are the two numbers whose sum and difference are indicated?

10. Express the product of two binomials having the common term $3a$. State the rule for finding such a product, then write the product of the two binomials you have written.

11. Find the product $(n - 3)(n + 7)$; $(2x + 5)(2x - 7)$.

12. Find the factors of $r^2 - r - 12$; of $n^2 - 8n + 15$.

13. The formula for the total surface of a right circular cone is $S = \pi r^2 + \pi rl$. Express the total surface of a cone the radius of whose base is $(2n - 1)$ feet and whose slant height, l, is 4 feet. Use the short form for performing the operations indicated.

CHAPTER IV

LINEAR EQUATIONS IN ONE UNKNOWN

29. Linear equations. It has been found very useful to be able to get new equations from given equations. If given a rectangle whose length is l and width w, the perimeter may be found from the equation $p = 2l + 2w$. But this equation does not tell us the value of l if we are given the values of w and p. By the use of certain truths called **axioms,** we can get a new equation having l alone in one member, the other member thus being the value of l.

Let us see how these new equations are obtained.

Axiom 1. *Equal numbers added to equal numbers give equal sums.*

Axiom 2. *Equal numbers subtracted from equal numbers give equal remainders.*

Axiom 3. *Equal numbers multiplied by equal numbers give equal products.*

Axiom 4. *Equal numbers divided by equal numbers give equal quotients.*

Axiom 5. *Any number may be substituted for its equal in an equation.*

By applying these axioms, equations are transformed into desired forms. The pupil must learn which axiom to apply to get the form desired.

From the equation $p = 2l + 2w$ it is desired to get an equation having l alone in the left member and only terms not containing l in the right member. To eliminate l from the right member subtract $2l$ from both members, applying Axiom 2. This gives the new equation $p - 2l = 2w$.

To eliminate p from the left member subtract p from each member, giving $-2l = 2w - p$.

To eliminate -2 from the left member divide each member by -2, applying Axiom 4, which gives $l = \frac{p}{2} - w$.

Transposing a term from one member to the other and changing its sign is an application of Axiom 2.

An equation containing the first and no higher power of the unknown is called a **linear equation.** Every linear equation in one unknown, x, can be put into the form $ax + b = 0$.

Rule. To solve a linear equation:

1. *Transpose to one member all terms containing the unknown, and to the other member all other terms.*
2. *Combine the terms containing the unknown number.*
3. *Divide each member by the coefficient of the unknown number.*

EXAMPLE. Solve for n, $3 + an = d - 4 + 2n$.

SOLUTION. Transposing, $an - 2n = d - 4 - 3$.

Combining terms, $(a - 2)n = d - 7$.

Dividing by $a - 2$, $n = \frac{d-7}{a-2}$.

Check this result by giving any values you choose to d and a, except $a = 2$ finding n from the last equation, then substituting these values in the original equation.

Exercise 33

Solve:

1. $2a - 3 = 4 - a$.
2. $5y - 2 - 3y + 7 = 0$.
3. $14r + 7 = 2r - 9 + 6r - 8$.
4. $5(n + 1) = 3(n + 7)$.
5. $2x - (5x + 4) = 8 - 6x$.
6. $5(a - 1) + 2(a + 1) = a + 15$.
7. $6 - 4(2 - 3x) = (7x - 5)$.
8. $7(r - 1) - 4(r - 2) = 3(r - 3) - 2(r - 4)$.
9. $(t - 1)\ (t + 1) = (t + 3)\ (t - 5)$.
10. $s = c + g$. Solve for c, for g.
11. $A = \frac{1}{2}ab$. Solve for a; for b.
12. $a = p + prt$. Solve for r; for t, for p.
13. $l = a + (n - 1)d$. Solve for a; for d; for n.

30. Using linear equations in solving problems. When the conditions of a problem are given in words, it is best, when possible, to translate them into an equation. The steps in solving such a problem are as follows:

1. Read the problem carefully.
2. Decide what are the unknown numbers involved.
3. Represent by a letter the unknown from which the other unknowns can be most easily found.
4. Express each of the other unknowns in terms of the one first selected.
5. From the conditions of the problem find two ways of expressing the same number or two numbers which are equal, thus forming an equation.
6. Solve the equation.
7. Test the result by seeing whether it satisfies the conditions as stated in the problem.

EXAMPLE. The perimeter of a rectangle is 32 feet. Its length is three times its width. Find the length of the rectangle.

SOLUTION.

1. Read the problem carefully.
2. The unknown numbers involved are the length and the width of the rectangle.
3. Represent by w the number of feet in the width, since the length, being 3 times the width, can easily be found.
4. Then $3w =$ the number of feet in the length.
5. The number of feet in the perimeter can be represented by $2w + 2 \cdot 3w$, also by 32 as given in the problem. Therefore $2w + 6w = 32$.
6. Solving the equation, $8w = 32$, we get $w = 4$, and $3w = 12$.
7. Since the length is 12 ft. and the width is 4 ft., the length is 3 times the width, and the perimeter is 32 ft.

Exercise 34

1. A tennis court is 6 feet more than twice as long as it is wide. Its perimeter is 228 feet. Find its length.
2. A rope 60 feet long is to be cut into two pieces so that one piece shall be 10 feet longer than the other. Find the length of each piece.

3. A two-by-four 18 ft. long is to be cut into two pieces. The longer piece is to be 3 ft. more than twice as long as the shorter. How long must each piece be?

4. Fred has $1.15 in nickels and dimes. He has two more nickels than dimes. How many of each has he?

5. Mary has counted the money in her bank and has put it back into the bank. Someone asks her how many dimes she has. She remembers that she has just as many dimes as nickels and as many cents as dimes and nickels together, also that the value of all her dimes, nickels, and cents is $1.87. Find out how many dimes she has.

6. A newsboy carrying the Tribune and the News finds that he has sold 117 papers for $3.09. The Tribune sells for 2¢, the News for 3¢. How many of each has he sold?

7. A rectangle and an equilateral triangle have the same perimeter. One side of the rectangle is the same length as one side of the triangle. How long are the sides of each figure, if the length of the rectangle is 4 ft. more than its width?

8. The average height of two boys is 4 ft. $7\frac{1}{2}$ in. One is $3\frac{1}{2}$ in. taller than the other. How tall is each?

9. The average of three numbers is 10. The third is double the first and the second is 5 less than the third. Find the numbers.

10. The average of seven consecutive integers is 15. Find them.

11. The sum of several numbers is 35. Their average is 5. How many numbers are there?

12. What sum of money put at interest at 6% will produce $420 interest in 7 years?

13. How long will it take $576 to amount to $648 at 5% interest?

14. During the World War the wages of a certain class of workmen were raised 115%. After the war their wages were reduced 50%, their wages then being 64.5¢ an hour. What had they received before the war?

15. A workman is now receiving $1.20 an hour, an increase of $166\frac{2}{3}\%$ over pre-war wages. By what per cent must his wages be reduced to give him pre-war wages?

16. A merchant has paid $8 each for some hats. At what price shall he mark them so that he can give a discount of $16\frac{2}{3}\%$ of the marked price and still make 25% of the cost?

17. A coat is marked to sell at $16. The merchant offers to give half the profit which that price would give him, and the customer takes the coat on those terms. The merchant finds that he has still made 30% on the cost of the coat. What did it cost him?

By the **speed, or rate of motion,** of a moving object we mean the distance the object moves in some unit of time. If the speed of an object is known, how can the distance it moves in a certain time be found? Make a formula for finding the distance, d, passed over by an object moving at the rate, r, for the time, t. How far does a train go in 3 hours 45 minutes, running at the rate of 40 miles an hour?

18. Two trains run in opposite directions from the same station. One runs at the rate of 25 miles an hour and the other at the rate of 45 miles an hour. In how many hours will they be 350 miles apart?

19. Two trains start from the same station and run in the same direction on parallel tracks, one at the rate of 30 miles an hour and the other at the rate of 36 miles an hour. In how many hours will they be 45 miles apart?

20. John has started to a town 30 miles away on his bicycle. He rides with a speed of 12 miles an hour. Ten minutes later his father starts after him in the automobile, running 30 miles an hour. How far from the town will his father overtake him?

21. At 3 o'clock the minute hand is just 15 minute spaces behind the hour hand. In how many minutes will the *minute hand* overtake the hour hand?

22. At what time between 4 and 5 o'clock are the hands of a clock together?

23. At what time between 2 and 3 o'clock are the hands of a clock in a straight line?

24. At what times between 7 o'clock and 8 o'clock are the hands of a clock at right angles to each other?

The quality of milk is determined by the amount of butterfat it contains. Four per cent milk means milk 4% of whose weight is butterfat. How much butterfat in 100 lb. of $3\frac{1}{2}$% milk? How much butterfat in a mixture of 30 lb. of 3% milk and 20 lb. of 4% milk? In a mixture of a pounds of 3% milk and b pounds of 4% milk?

25. A farmer wishes to mix a quantity of 3% milk and a quantity of $4\frac{1}{2}$% milk to get 80 lb. of $3\frac{1}{2}$% milk. How much of each kind shall he use?

Suggestion. Find how much butterfat in p pounds of 3% milk and in $(80-p)$ pounds of $4\frac{1}{2}$% milk. This must be the number of pounds of butterfat in the 80 pounds of $3\frac{1}{2}$% milk.

26. To get 100 lb. of 4.5% milk a dairyman mixes $3\frac{1}{2}$% milk with 5% milk. How much of each quality must he use?

27. How many pounds of 40% cream must be mixed with 15 lb. of 4% milk to get 25% cream?

28. How many pounds of 5% milk must be used with 60 lb. of 35% cream to get 30% cream?

29. How many pounds of a% cream must be mixed with p pounds of c% cream to get b% cream? The answer to this problem is a formula by which all such problems may be solved. Use it to solve No. 27 and No. 28.

30. Alcohol is never pure but always contains a certain amount of water. It is said to be 95% pure if 95% of it is pure alcohol. How many pints of 80% alcohol mixed with 12 pints of 95% alcohol will give a mixture 90% alcohol?

31. How many gallons of water would be required to dilute 36 gallons of 92% alcohol to 40% alcohol?

Exercise 35. Review Questions

1. Write a general odd number. Show that the square of any odd number is an odd number.

2. Find the area of an equilateral triangle whose side is 6 in. Use the formula.

3. The digits of a four-place number from left to right are x, 0, y, and z. What is the number?

4. Which of the following numbers are divisible by 2? By 3? 87,431; 63,529; 3,682,152; 76,494; 81,252; 76,234,902; 419,263,045.

5. Simplify $(2a^2b-3ab^2-a^2)-(-3a^2+b^2-5ab^2)+(2b^2-3a^2b)$.

6. What is the rth term of an arithmetic progression whose second term is b and common difference h?

7. $(-1)^{2n+1} = ?$ $(-2)^2 + (-2^2) = ?$

8. Simplify $(4a + 3x)^2$.

9. What are the factors of $169 - x^2$? Of $s^2 - 2s - 63$?

10. A cone and a cylinder have the same radius, r, and the same altitude, a. Find the sum of their volumes, also the sum of their curved surfaces.

11. Express as a formula the interest, compounded annually, on P dollars for n years at $r\%$.

12. The last term of an arithmetic progression is -15, the first term is 12, and the common difference is -3. How many terms in the series?

13. What are the steps in solving a problem by the use of an equation?

14. Use the formula for n, found by solving the equation, $l = a + (n - 1)d$, to find how many odd numbers there are from 17 to 45, both inclusive.

15. The first and the last terms of an arithmetic series of 9 terms are 4 and 28. What is the common difference?

CHAPTER V

FRACTIONS

31. Fractions. In algebra a **fraction,** as $\frac{a}{b}$, means the quotient of a divided by b.

In the fraction $\frac{a}{b}$, a and b are called the **terms** of the fraction; a is called the **numerator** and b the **denominator.**

The rules for fractions used in arithmetic apply also in algebra. Can you tell how to

(a) Reduce fractions to lowest terms?
(b) Reduce fractions to a common denominator?
(c) Add fractions?
(d) Subtract fractions?
(e) Multiply a fraction by a fraction?
(f) Divide a fraction by a fraction?

If you know how to do these things you should have little difficulty in this chapter.

Principle. *The numerator and denominator of a fraction may be multiplied or divided by the same number without changing the value of the fraction.*

Thus, $$\frac{9}{12} = \frac{9 \div 3}{12 \div 3} = \frac{3}{4}.$$

Also, $$\frac{5}{6} = \frac{3 \cdot 5}{3 \cdot 6} = \frac{15}{18}.$$

Similarly, $$\frac{3xy}{5ab} = \frac{3xy \cdot 2a}{5ab \cdot 2a} = \frac{6axy}{10a^2b};$$

and $$\frac{6a^3b}{9a^2b^2} = \frac{6a^3b \div 3a^2b}{9a^2b^2 \div 3a^2b} = \frac{2a}{3b}.$$

32. Reduction to lowest terms. A fraction is in its **lowest terms** when the numerator and denominator have no common factor except 1.

The fraction $\frac{7}{8}$ is in its lowest terms, but $\frac{6}{8}$ is not, since 6 and 8 have the common factor 2.

Rule. To reduce a fraction to its lowest terms divide the numerator and the denominator by the factors common to both.

Exercise 36

Reduce to lowest terms:

1. $\frac{6}{8}$.

2. $\frac{10}{12}$.

3. $\frac{15}{20}$.

4. $\frac{25}{30}$.

5. $\frac{36}{48}$.

6. $\frac{100}{120}$.

7. $\dfrac{4a}{12}$.

8. $\dfrac{12x}{6}$.

9. $\dfrac{3x}{15y}$.

10. $\dfrac{ab}{a^2}$.

11. $\dfrac{x^2y^2}{3x^3y}$.

12. $\dfrac{2abc}{abc}$.

13. $\dfrac{14m^2n}{7m^3n^4}$.

14. $\dfrac{a(a-b)}{a^2}$.

15. $\dfrac{x-y}{(x-y)^2}$,

16. $\dfrac{a^2+ab}{a}$.

17. $\dfrac{r^2+r}{r+1}$.

18. $\dfrac{2a+4}{6}$.

19. $\dfrac{6x}{9x+12}$.

20. $\dfrac{36x^2y^3}{12x^4y}$.

21. $\dfrac{2+2x}{2}$.

22. $\dfrac{3}{6x+3}$.

23. $\dfrac{2}{2+x}$.

24. $\dfrac{5y-10z}{5y+10z}$.

25. $\dfrac{12x+4}{3x+1}$.

26. $\dfrac{\frac{4}{3}\pi r^3}{4\pi r^2}$.

27. $\dfrac{\pi r^2}{2\pi r}$.

28. $\dfrac{a+b}{a-b}$.

29. $\dfrac{a^2-b^2}{a+b}$.

30. $\dfrac{a+2}{b+2}$.

31. $\dfrac{a^2-2ab+b^2}{a^2-b^2}$.

32. $\dfrac{x^2+6x+8}{x^2+4x+4}$.

33. $\dfrac{m^2+mn}{m+n}$.

34. $\dfrac{2ab}{2a+2b}$.

35. $\dfrac{m^2n^2}{m^2+mn}$.

36. $\dfrac{15a^3b^2c}{40abc^2}$.

37. $\dfrac{3x^2+6x}{x^2-4}$.

38. $\dfrac{x^2-7x+10}{x^2-2x-15}$.

39. $\dfrac{x^3+3x^2}{x^2+x-6}$.

40. $\dfrac{a^2+ab}{a^2-ab}$.

41. $\dfrac{3a^2b+15ab^2}{4ab+20b^2}$.

42. $\dfrac{(x+y)^3}{(x+y)^2}$.

33. Signs in fractions. In any fraction there are three signs to be considered, the sign of the numerator, the sign of the denominator, and the sign of the fraction, which is written just before the line between the numerator and the denominator.

Thus, the fraction, $+\frac{-4}{+7}$, has the three signs written.

Since a fraction is an indicated division, numerator, denominator, and fraction mean dividend, divisor, and quotient, respectively. Hence the law of signs for division holds for fractions, and we have $\frac{-5}{6} = -\frac{5}{6}$; $\frac{5}{-6} = -\frac{5}{6}$; $\frac{-5}{-6} = \frac{5}{6}$; and $\frac{+5}{+6} = \frac{5}{6}$.

If we use all three signs in each case, we have,

$$+\frac{-5}{+6} = -\frac{+5}{+6};\ +\frac{+5}{-6} = -\frac{+5}{+6};\ +\frac{-5}{-6} = +\frac{+5}{+6}.$$

These examples illustrate the principle:

Any two of the three signs in a fraction may be changed without changing the value of the fraction.

In dealing with fractions it is usually most convenient to have the signs of both numerator and denominator positive.

Much use is made of such relations as these:

$$-\frac{-5}{+6} = +\frac{+5}{+6};\ -\frac{+5}{-6} = +\frac{+5}{+6};\ +\frac{-5}{+6} = -\frac{+5}{+6}.$$

The sign of the fraction is changed by changing the sign of the numerator or of the denominator alone.

The pupil should remember that changing the sign of a number is equivalent to **multiplying or dividing it by 1.** Hence in changing the sign of a numerator or denominator when it is a polynomial, care must be taken to **change the sign of each term of the polynomial, including the first.**

Thus we get

$$-\frac{-a+b-c}{2a-4}=\frac{a-b+c}{2a-4},$$

by changing the sign of the numerator and the sign of the fraction.

34. Common errors. Fractions are reduced to their lowest terms by dividing both numerator and denominator by factors that are common to both numerator and denominator. It is a common error to remove common **terms** instead of common factors. Thus, in the fraction $\frac{2+a}{2+b}$ it is an error to remove the term 2 and to say that the result, $\frac{a}{b}$, equals the original fraction, as 2 is not a common factor in the numerator and the denominator.

In the fraction, $\frac{ab}{abc}$, all of the factors in the numerator are also factors of the denominator. Hence when the fraction is reduced to its lowest terms we have $\frac{ab}{abc}=\frac{1}{c}$. The error is often made of writing 0 for the numerator in such a case. Do you see any cause for this error?

Exercise 37

Change the following into equivalent fractions having positive numerators and positive denominators:

1. $\frac{-5}{6}$.
2. $\frac{4}{-9}$.
3. $\frac{-a}{-b}$.
4. $-\frac{c}{d}$.
5. $-\frac{-x}{2}$.
6. $-\frac{-a}{-u}$.

Reduce the following to equivalent fractions having the first term in both numerator and denominator positive:

7 $\dfrac{a+b}{-5}.$

8. $-\frac{2}{3}.$

9. $-\dfrac{c-a}{-b}.$

10. $\dfrac{x-y+z}{-r-s}.$

11. $\dfrac{2+a+b}{-8}.$

12. $-\dfrac{-m+n}{m-n}.$

Reduce to lowest terms:

13. $\dfrac{a}{-a}.$

14. $\dfrac{-12an}{4a^2}.$

15. $\dfrac{a-1}{a}.$

16. $\dfrac{2+2a}{2}.$

17. $\dfrac{(x-y)^2}{(x-y)^4}.$

18. $\dfrac{a-4}{4-a}.$

19. $\dfrac{m-n}{n-m}.$

20. $\dfrac{8a-6}{2a}.$

21. $\dfrac{x(x-y)}{xy(x-y)^2}.$

22. $\dfrac{3ab}{12ab^3}.$

23. $\dfrac{6a^3b^2c}{9a^2b^3c}.$

24. $\dfrac{12m^2n^3}{20m^3n^2}.$

25. $\dfrac{a^2-b^2}{a+b}.$

26. $\dfrac{a^2+2ab+b^2}{a^2-b^2}.$

27. $\dfrac{x-3}{x^2-7x+12}.$

28. $\dfrac{2t-2}{t^2-1}.$

29. $\dfrac{6+3x}{9+3x}.$

30. $\dfrac{m^2-m-6}{m^2-9}.$

Using the values given, evaluate the following fractions. Compute results correct to .01.

31. $\dfrac{2MN}{M+N}.$ $M = 20,\ N = 35.$

32. $\dfrac{2a^2+2b^2-c^2}{4}.$ $a = 10,\ b = 25,\ c = 9.$

33. $\dfrac{Q}{4\pi r^2}.$ $Q = 156,\ r = 12,\ \pi = 3\frac{1}{7}.$

34. $\frac{1}{2}\left(\dfrac{a}{n-a} - \dfrac{b}{n-b}\right).$ $a = .12,\ b = .02,\ n = 4.2.$

35. $\dfrac{1}{t}\left(\dfrac{T^2}{K^2} - 1\right).$ $T = 12,\ K = 2.4,\ t = .04.$

36. $\frac{4}{3}\left(\dfrac{m+n}{m} - \dfrac{m-n}{n}\right).$ $m = 20,\ n = 16.$

35. Lowest common multiple. A number is said to be a **multiple** of any of its factors. Thus, 6 is a multiple of 3 and of 2; $3ax^2$ is a multiple of 3, a, x, x^2, $3a$, $3ax$, $3x^2$, ax, and ax^2.

A **common multiple** of two or more algebraic expressions is an expression that is a multiple of each of them. Thus, 12 is a common multiple of 4 and 6, $7x^2y$ is a common multiple of x^2 and y.

A given set of expressions, as, $5ab$, $2x$, and 9, has an unlimited number of common multiples. Can you name 6 common multiples of these expressions? One of these is of much importance. It is called the **lowest common multiple.** *The lowest common multiple of two or more expressions is the product of all their common prime factors, each factor being used the greatest number of times that it occurs in any of the expressions.* As an example, consider the expressions $4ax$, a, $10x^2$, and 7. The different prime factors of these expressions are 2, 5, 7, a, and x. The factors a, 5, and 7 occur only once in any expression. The factor 2 occurs twice in $4ax$ and x occurs twice in $10x^2$. The L.C.M. is then $2\cdot 2\cdot 5\cdot 7\cdot a\cdot x\cdot x = 140ax^2$.

The method of finding the L.C.M. is further illustrated in the following examples.

EXAMPLE 1. Find the L.C.M. of 20, 48, and 35.

SOLUTION.

$$
\begin{aligned}
20 &= 2\cdot 2\cdot 5.\\
48 &= 2\cdot 2\cdot 2\cdot 2\cdot 3.\\
35 &= 5\cdot 7.\\
\text{L.C.M.} &= 2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 5\cdot 7 = 1680.
\end{aligned}
$$

EXAMPLE 2. Find the L.C.M. of $6ax^2$, $5ax$, $a^2 + ab$, and $a^2 + 3ab + 2b^2$.

SOLUTION.

$$
\begin{aligned}
6ax^2 &= 2\cdot 3\cdot a\cdot x\cdot x.\\
5ax &= 5\cdot a\cdot x.\\
a^2 + ab &= a(a + b).\\
a^2 + 3ab + 2b^2 &= (a + b)\ (a\ \ 2b)\\
\text{L.C.M.} &= 2\cdot 3\cdot 5\cdot a\cdot x\cdot x\cdot (\ a + b)(a\ 2b) = 30ax^2(a + b)(a + 2b).
\end{aligned}
$$

Exercise 38

Find the L.C.M. of the following sets of expressions:

1. xy, xy^2, and x^2y.
2. $12a^2$, $3ab$, and $10b^3$.
3. abc, axy, and bcx.
4. $30ab^2$, $30a^2b^2$, and $48a$.
5. a, $a + b$, and b.
6. $a + b$, $a - b$, and $a^2 - b^2$.
7. $x + 1$ and $x + 2$.
8. $a - b$ and $(a - b)^2$.
9. a, b, c, and d.
10. $x^2 + xy$ and $x + y$.
11. 2, a, and $2a + b$.
12. $r + w$ and $r - w$.
13. 1, 5 and $2a$.
14. 3, a, and $6a + b$.
15. $a^2 - b^2$ and $a^2 + 2ab + b^2$.
16. $r^2 + 5r + 6$, $r + 2$, and $r + 3$.

36. Addition and subtraction. As in arithmetic, fractions to be added or subtracted must have a common denominator. Of the unlimited number of common denominators that may be found for a set of fractions, the least common denominator is usually chosen, as it is the most convenient. The **lowest common denominator** of the fractions is the lowest common multiple of the denominators.

Exercise 39

Solve as many as possible orally.

1. $\frac{5}{9} + \frac{2}{9}$.
2. $\frac{5}{13} + \frac{2}{13}$.
3. $\frac{5}{a} + \frac{2}{a}$.
4. $\frac{5c}{a} + \frac{2c}{a}$.
5. $\frac{x}{a} + \frac{y}{a}$.
6. $\frac{5}{9} - \frac{2}{9}$.
7. $\frac{5}{13} - \frac{2}{13}$.
8. $\frac{5}{a} - \frac{2}{a}$.
9. $\frac{5c}{a} - \frac{2c}{a}$.
10. $\frac{x}{a} - \frac{y}{a}$.
11. $\frac{6}{a+1} + \frac{4}{a+1}$.

12. $\dfrac{6}{a+1} - \dfrac{4}{a+1}$.

13. $\dfrac{a}{a+1} + \dfrac{1}{a+1}$.

14. $\dfrac{x}{x+y} - \dfrac{y}{x+y}$.

15. $\dfrac{r}{c+d} - \dfrac{r-2}{c+d}$.

16. $\dfrac{m+n}{r} + \dfrac{m-n}{r}$.

17. $\dfrac{2x-5}{n} - \dfrac{6}{n}$.

18. $\dfrac{s+t}{v} + \dfrac{s-t}{v}$.

19. $2 + \frac{3}{5}$.

20. $5 + \frac{7}{9}$.

21. $a + \frac{3}{5}$.

22. $b + \frac{1}{4}$.

23. $x + \dfrac{1}{y}$.

24. $m - \dfrac{k}{n}$.

25. $c + \dfrac{1}{c}$.

26. $1 - \dfrac{1}{c}$.

27. $\dfrac{2}{a} - a$.

28. $2 + a + \frac{1}{2}$.

29. $x + y - \frac{2}{3}$.

30. $a + b + \dfrac{1}{c}$.

31. $\dfrac{s+t}{v} - \dfrac{s-t}{v}$.

Suggestion. The second numerator, $s - t$, is to be subtracted from $s + t$; therefore the sign of each term of $s - t$ must be changed.

32. $\dfrac{m}{c} - \dfrac{m-n}{c}$.

33. $\dfrac{2x-y}{a} - \dfrac{2x+y}{a}$.

34. $\dfrac{r+2}{m-n} - \dfrac{2r+3}{m-n}$.

It is convenient to have written work arranged in such a form as the following:

EXAMPLE 1. Perform the addition: $\dfrac{a}{b} + \dfrac{c}{d}$.

SOLUTION. The lowest common denominator is bd.

Then
$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}.$$

EXAMPLE 2. Simplify: $\dfrac{3}{4} + \dfrac{2a}{3b} - \dfrac{3}{2ab}$.

SOLUTION.
$$\frac{3}{4} + \frac{2a}{3b} - \frac{3}{2ab} = \frac{9ab}{12ab} + \frac{8a^2}{12ab} - \frac{18}{12ab} = \frac{9ab + 8a^2 - 18}{12ab}.$$

Example 3. Simplify: $\frac{5}{6a} + \frac{3a}{a-b} - \frac{2b}{a+b}.$

Solution.

$$\frac{5}{6a} + \frac{3a}{a-b} - \frac{2a}{a+b} = \frac{5(a-b)\ (a+b)}{6a(a-b)\ (a+b)} + \frac{3a \cdot 6a(a+b)}{6a(a-b)\ (a+b)} - \frac{2a \cdot 6a \cdot (a-b)}{6a \cdot (a-b)\ (a+b)}$$

$$= \frac{5(a-b)\ (a+b) + 3a \cdot 6a(a+b) - 2a \cdot 6a(a-b)}{6a(a-b)\ (a+b)}$$

$$= \frac{5a^2 - 5b^2 + 18a^3 + 18a^2b - 12a^3 + 12a^2b}{6a(a-b)\ (a+b)}$$

$$= \frac{6a^3 + 5a^2 + 30a^2b - 5b^2}{6a(a-b)\ (a+b)}.$$

How is the lowest common denominator found in Example 2? Tell how each numerator is found when the fractions are reduced to a common denominator.

Exercise 40

Perform the following additions and subtractions:

1. $\frac{1}{a} + \frac{1}{b}.$
2. $\frac{2}{x} - \frac{3}{y}.$
3. $\frac{n}{xy} + \frac{m}{yz}.$
4. $\frac{3}{4} + \frac{a}{a+1}.$
5. $\frac{1}{a} + \frac{2}{b} - \frac{3}{c}.$
6. $\frac{m-2}{4} - \frac{m+3}{5}.$
7. $\frac{r+s}{2} + \frac{r-s}{6}.$
8. $\frac{a}{8} - \frac{2a}{8} + \frac{3}{8}.$
9. $\frac{a-1}{a} + \frac{a^2-1}{a^2} + \frac{a^3-1}{a^3}.$
10. $\frac{2}{a} + \frac{a}{2} - \frac{1}{a}.$
11. $a + \frac{3}{a}.$
12. $5a - \frac{4a+3}{2a}.$
13. $a - 1 + \frac{1}{a-1}.$
14. $\frac{3x}{x+2} - \frac{4}{x-2}.$

15. $\frac{y+3}{2y}-\frac{y-5}{7}.$

16. $\frac{a-1}{a+1}-\frac{a+1}{a-1}.$

17. $\frac{m}{n}+\frac{n}{m}+\frac{1}{m}+\frac{1}{n}.$

18. $12+\frac{a+6}{3a}-\frac{5}{8}.$

19. $\frac{6m}{m^2-1}+\frac{m+1}{m-1}.$

20. $3-a+b-\frac{1}{2}.$

21. $\frac{5}{x+3}+\frac{3}{x+4}.$

22. $\frac{4}{x+1}-\frac{3}{x+4}.$

23. $\frac{2a}{3b-2a}-\frac{a}{2a-3b}.$

24. $\frac{x}{x-y}-\frac{y}{x+y}.$

25. $\frac{x}{x+1}-\frac{x^2}{x^2-1}.$

26. $\frac{r}{s(r-s)}+\frac{1}{r-s}.$

27. $\frac{4}{6t+15}-\frac{5}{8t+20}.$

28. $\frac{m-n}{2m+2n}-\frac{m+n}{3m-3n}.$

29. Find the perimeter of a rectangle whose length is $\frac{3x-1}{x+2}$ and its width $\frac{3x+1}{x-2}.$

30. How much longer is the perimeter of a rectangle $\frac{a}{a-b}$ feet long by $\frac{b}{a+b}$ feet wide than the perimeter of an equilateral triangle $\frac{ab}{a^2-b^2}$ feet on each side?

37. Multiplication of fractions. As in arithmetic the product of any number of fractions is the fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators.

Thus, in arithmetic,

$$\frac{3}{4}\cdot\frac{\cancel{5}}{\underset{2}{\cancel{12}}}\cdot\frac{\cancel{6}}{\underset{2}{\cancel{10}}}=\frac{3}{16}.$$

In algebra,

$$\frac{\cancel{a+1}}{\cancel{b}}\cdot\frac{a^2\cancel{b}}{\underset{xy^2}{\cancel{x^2y^2}}}\cdot\frac{\cancel{x}}{\underset{a-1}{\cancel{a^2-1}}}=\frac{a^2}{xy^2(a-1)}.$$

Both numerator and denominator should be divided by all of their common prime factors.

Expressions in integral form may be considered as fractions with a denominator unity. Thus, $2x + 1 = \frac{2x+1}{1}$.

Exercise 41

Perform the following multiplications and reduce the products to their lowest terms:

1. $\frac{4}{5}\cdot\frac{2}{3}.$
2. $\frac{3}{8}\cdot\frac{2}{5}.$
3. $\frac{a}{b}\cdot\frac{c}{d}.$
4. $\frac{r}{5}\cdot\frac{7}{8}.$
5. $4\cdot\frac{2}{3}.$
6. $a\cdot\frac{x}{y}.$
7. $m\cdot\frac{2}{m}.$
8. $\frac{2a}{3b}\cdot b.$
9. $5x\cdot\frac{a}{x}.$
10. $\frac{a}{2}\cdot\frac{2}{a}.$
11. $\frac{5}{21}\cdot\frac{7}{20}.$
12. $\frac{15}{28}\cdot\frac{16}{25}.$
13. $\frac{3ab}{4ax}\cdot\frac{8x^2}{9b^2c}.$
14. $\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{d}.$
15. $\frac{5b^2}{2a^2}\cdot\frac{8ab}{25c}\cdot\frac{7a^2}{12b}.$
16. $\frac{3^2}{5^3}\cdot\frac{10^2}{6}.$
17. $\frac{2^2}{3^2}\cdot\frac{3^3}{4^3}\cdot\frac{4^4}{5^4}.$
18. $6\cdot\frac{a}{2b}\cdot\frac{5b^2}{3a^2}.$
19. $4a\cdot\frac{ax}{6m^2}\cdot\frac{3mn^2}{5a^3}.$
20. $\frac{x-y}{x}\cdot\frac{x}{x-y}.$
21. $\frac{x^2-y^2}{x^3}\cdot\frac{x}{x+y}.$
22. $a\cdot\frac{m}{b}\cdot b.$
23. $\frac{m+n}{mn}\cdot\frac{m-n}{m^2+2mn+n^2}.$
24. $\frac{x^2-xy-2y^2}{x^2-y^2}\cdot\frac{x^2+xy}{x^2-4y^2}.$
25. $\frac{a^2+3a+2}{a-3}\cdot\frac{a^2-4a+3}{a^2-1}.$
26. $\frac{x^2-xy}{x}\cdot\frac{x^3}{x^2-y^2}.$

38. Division of fractions. The method of dividing one fraction by another is the same in algebra as in arithmetic. By the definition, the divisor times the quotient equals the dividend.

Hence, to divide $\frac{3}{5}$ by $\frac{4}{7}$ is to find a number q such that

$$\frac{4}{7}q = \frac{3}{5}.$$

Multiplying both members of the equation by $\frac{7}{4}$, we have

$$q = \frac{3}{5} \times \frac{7}{4} = \frac{21}{20}.$$

This example illustrates the

Rule. *To divide by a fraction multiply the dividend by the inverted divisor.*

Exercise 42

Perform the indicated operations and simplify the results:

1. $\frac{2}{3} \div \frac{3}{4}$.

2. $\frac{1}{3} \div \frac{1}{4}$.

3. $\frac{5}{6} \div \frac{7}{8}$.

4. $\dfrac{a}{b} \div \dfrac{c}{d}$.

5. $\dfrac{1}{b} \div \dfrac{1}{c}$.

6. $\dfrac{ab}{c} \div c$.

7. $\dfrac{2a}{bc} \div \dfrac{4a}{cd}$.

8. $\dfrac{5x^2}{12a^2} \div \dfrac{10x}{3ab}$.

9. $\dfrac{5}{7} \div \dfrac{a}{b}$.

10. $\dfrac{20ab}{9m^2n} \div \dfrac{5a^2}{6mn}$.

11. $\dfrac{m^2 - n^2}{mn} \div \dfrac{m - n}{n^2}$.

12. $\dfrac{x^2 - 9y^2}{x + y} \div \dfrac{x - 3y}{x^2 + 2xy + y^2}$.

13. $\dfrac{abc}{a - 1} \div ab$.

14. $\dfrac{a}{b} \div a$.

15. $a \div \dfrac{a}{b}$.

16. $\dfrac{x}{y} \div xy$.

17. $\dfrac{m}{n} \div (m - n)$.

18. $(r + s) \div \dfrac{r}{s}$.

19. $\left(1 + \dfrac{1}{a}\right) \div a$.

20. $b \div \left(b - \dfrac{1}{b}\right)$.

Exercise 43

Combine into a single fraction:

1. $\frac{1}{x} - \frac{3}{x}$.
2. $\frac{a}{3} + \frac{a}{4}$.
3. $\frac{1}{a} + 2$.
4. $6 - \frac{b}{c}$.
5. $\frac{1}{x} + x$.
6. $5 - \frac{a+1}{5}$.
7. $\frac{c}{d} + \frac{a}{b}$.
8. $7 - \frac{1}{x} + 3$.
9. $1 - \frac{a}{a-1}$.
10. $a + \frac{3}{5}$.

Perform the indicated operations and simplify the results:

11. $y \cdot \frac{x}{y}$.
12. $m \cdot \frac{m}{n}$.
13. $\frac{5}{6} + \frac{2}{6}$.
14. $\frac{5}{a} + \frac{2}{a}$.
15. $\frac{a+4}{a} \div 2$.
16. $\frac{5}{6} \div 12$.
17. $\frac{10+15}{2+3}$.
18. $\frac{6 \cdot 12}{6}$.
19. $\frac{a^2+1}{x} \div \frac{a^2}{x}$.
20. $\frac{m+2}{2} + 2$.
21. $(b-1) \cdot \frac{b+2}{b-1}$.
22. $(a-3) \cdot \frac{a+4}{(a+2)(a-3)}$.
23. $(x+y)(x-y) \cdot \frac{1}{x+y}$.
24. $(x-y) \div \frac{1}{x-y}$.
25. $(a^2 - 5a + 6) \cdot \frac{a-2}{a-3}$.
26. $\frac{1}{x+2} \cdot \quad (x^2 + 4x + 4)$.
27. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.
28. $abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$.

39. Complex fractions. A **complex fraction** is one which has a fraction in one or both of its terms.

A complex fraction can be reduced to a simple fraction by multiplying both of its terms by their least common denominator.

EXAMPLE 1. Reduce $\frac{\frac{2}{3}}{4\frac{1}{2}}$ to a simple fraction.

SOLUTION.
$$\frac{\frac{2}{3}}{4\frac{1}{2}} = \frac{6 \times \frac{2}{3}}{6 \times 4\frac{1}{2}} = \frac{4}{27}.$$

EXAMPLE 2. Reduce $\dfrac{4 + \frac{a}{b}}{c + \frac{1}{2}}$ to a simple fraction.

SOLUTION.
$$\frac{4 + \frac{a}{b}}{c + \frac{1}{2}} = \frac{2b\left(4 + \frac{a}{b}\right)}{2b\left(c + \frac{1}{2}\right)} = \frac{8b + 2a}{2bc + b}.$$

EXAMPLE 3. Reduce $\dfrac{\frac{a}{m+3}}{\frac{2ab}{c}}$ to a simple fraction.

SOLUTION. This case should be treated as the division of one fraction by another.

Then,
$$\frac{\frac{a}{m+3}}{\frac{2ab}{c}} = \frac{a}{m+3} \div \frac{2ab}{c}$$
$$= \frac{\cancel{a}}{m+3} \times \frac{c}{2\cancel{a}b}$$
$$= \frac{c}{2b(m+3)}.$$

Exercise 44

Reduce to simple fractions in their lowest terms:

1. $\dfrac{\frac{5}{6}}{\frac{3}{4}}$.

2. $\dfrac{\frac{a}{b}}{\frac{c}{d}}$.

3. $\dfrac{\frac{m}{n}}{x}$.

4. $\dfrac{z}{\frac{5}{7}}$.

5. $\dfrac{a}{5\frac{1}{2}}$.

6. $\dfrac{\frac{1}{2}gt^2}{6}$.

7. $\dfrac{\frac{1}{4}d^2}{10}$.

8. $\dfrac{1}{\dfrac{1}{a}+\dfrac{1}{b}}$.

9. $\dfrac{a+\frac{1}{2}}{3}$.

10. $\dfrac{x+\dfrac{1}{x}}{x-\dfrac{1}{x}}$.

11. $\dfrac{a+b}{\dfrac{1}{a+b}}$.

12. $\dfrac{A}{\frac{22}{7}}$.

13. 4 is what part of 5? 8 is what part of 13?

14. a is what part of b? x is what part of 7?

15. How do you find what part a number a is of a number b?

16. A newsboy's route is $1\frac{3}{8}$ miles. He has gone $\frac{5}{8}$ of a mile. What part of the route has he yet to go?

17. Two boys dig a patch of potatoes. It is estimated that one digs $2\frac{1}{2}$ acres and the other $1\frac{3}{4}$ acres. What part of the pay should each receive?

18. Three boys do a job of work. The first works 3 hours, the second works $4\frac{1}{2}$ hours, and the third $3\frac{1}{4}$ hours. What part of the pay should each receive?

19. Answer the same question if the first boy works p hours, the second q hours, and the third r hours.

20. It is desired to compare the sizes of two varieties of steel balls. When 12 of the first kind are dropped into a cylindrical vessel partly filled with water the level of the water is raised 3 inches; when 20 of the second kind are dropped into the same vessel the level of the water is raised $2\frac{1}{2}$ inches. Which kind is larger? How many times as large as the other?

21. If p of the larger kind raise the level of the water a inches, and q of the smaller kind raise the level of the water b inches, one of the larger is how many times as large as one of the smaller?

40. Work problems. Many problems about doing a piece of work give rise to the same formula. This formula involves fractions.

Exercise 45

1. What part of a piece of work can A do in 1 day if he can do all of the work (*a*) in 2 days; (*b*) in 7 days; (*c*) in $2\frac{1}{2}$ days; (*d*) in $4\frac{3}{4}$ days; (*e*) in r days; (*f*) in $p+q$ days; (*g*) in $t+4$ days? Tell in each case what part he can do in 2 days; in d days.

2. A can do a piece of work in m days and B in n days. What part of the work can each do in 1 day? What part can they both together do in 1 day? Let x be the number of days that it will take them together to do all of the work. Then by using x express the part of the work that they can do together in 1 day. You should now be able to get an equation involving m, n, and x.

SOLUTION. $\frac{1}{m}$ = the part of the work that A can do in 1 day.

$\frac{1}{n}$ = the part of the work that B can do in 1 day.

$\frac{1}{x}$ = the part of the work that they can both do in 1 day.

Hence, $$\frac{1}{x}=\frac{1}{m}+\frac{1}{n}=\frac{m+n}{mn}.$$

Then, $$x=\frac{mn}{m+n}.$$

Can you translate this formula into a rule?

Use this formula to find how long it will take A and B together to do the work if A can do it in 3 days and B in 2 days; also if A can do the work in 7 days and B in 5 days; A in 10 days and B in 8 days; A in $2\frac{1}{2}$ days and B in $4\frac{1}{3}$ days.

3. Make a formula that gives the number of days, x, that A, B and C will require to do a piece of work, working together, if A can do the work in m days, B in n days, and C in p days.

4. Use the formula of the preceding exercise to find how many days A, B, and C will require working together if A requires 4 days, B $5\frac{1}{2}$ days, and C 3 days.

5. A can drive from Charleston to Paris in 50 minutes, and B can drive from Paris to Charleston in 1 hour. How many minutes will it take them to meet if A starts from Charleston and B from Paris?

6. One tap can fill a bathtub in 10 minutes and another in 12 minutes. How long will it take to fill the tub if both taps are running?

7. One tap can fill a bathtub in t_1 minutes and another in t_2 minutes. Make a formula that will give the number, T, of minutes required to fill the bathtub if both taps are running.

8. A can walk around a circular track in 5 minutes, and B can walk around it in 6 minutes. What fraction of the length of the track can each walk in one minute? If they start together and walk in the same direction, what fraction of the length of the track will A gain on B in one minute? If they continue walking in the same direction, in how many minutes will A overtake B?

9. Answer the same questions if A can walk around the track in t_1 minutes, and B in t_2 minutes. Make a formula that gives the number of minutes, T, required for A to overtake B.

10. Use this formula to find the number of minutes required for A to overtake B, if A walks around in 10 min., and B in 8 min.

Exercise 46. Review of Fractions

In the first three exercises assume that the letters represent positive integers.

1. Which is greater, $\frac{1}{a}$ or $\frac{1}{a+b}$? $\frac{5}{7}$ or $\frac{5}{8}$? $\frac{a}{b}$ or $\frac{a}{b+2}$? What is the effect upon the value of a fraction of increasing the denominator?

2. Which is greater, $\frac{3}{10}$ or $\frac{7}{10}$? $\frac{7}{a}$ or $\frac{12}{a}$? $\frac{a}{b}$ or $\frac{a+1}{b}$? What is the effect upon the value of a fraction of increasing the numerator?

3. Suppose that a is less than b. What kind of a fraction is $\frac{a}{b}$? Let c be added to each term of this fraction. What is the resulting fraction? Which is greater, $\frac{a}{b}$ or $\frac{a+c}{b+c}$? What is the effect upon the value of a proper fraction of adding the same number to both terms?

HINT. To find which of the two fractions $\frac{a}{b}$ or $\frac{a+c}{b+c}$ is greater, reduce them to a common denominator.

4. What is the effect upon the value of an improper fraction of adding the same number to both terms?

5. The side of a square sheet of metal is a. When the sheet is heated it expands until the edge is $a+\frac{1}{t}$. What is the area then? What is the increase in area?

6. The number of units in the perimeter of a square of side a is what part of the number of units in its area?

7. The number of units in the circumference of a circle of radius r is what part of the number of units in its area?

8. The number of units in the volume of a cube of edge e is what fraction of the number of units in the area of its surface?

9. Reduce the formula $M = \dfrac{m+n}{\dfrac{1}{m}+\dfrac{1}{n}}$ to its simplest form for purposes of computation.

SOLUTION. This formula will be in a simpler form if it is reduced to a simple fraction. Multiplying both numerator and denominator by mn we get

$$M = \frac{mn(m+n)}{m+n} = mn.$$

Reduce the following formulas to simpler forms for purposes of calculation:

10. $K = rs\left(\dfrac{1}{r}+\dfrac{1}{s}\right).$

11. $T = \dfrac{4}{t} - \dfrac{t}{4}.$

12. $A = \dfrac{1}{a} + \dfrac{1}{b} - \dfrac{2}{ab}.$

13. $S = \dfrac{a}{bc} + \dfrac{b}{ac} + \dfrac{c}{ab}.$

14. $C = \dfrac{1+\dfrac{1}{c}}{1-\dfrac{1}{c}}.$

15. $N = \dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{1}{a}-\dfrac{1}{b}}.$

16. $E = \dfrac{a-\frac{1}{2}}{b+\frac{3}{4}}.$

17. Show that $\dfrac{1}{1-a} = 1 + \dfrac{a}{1-a}.$

18. Show that $\dfrac{1}{1+a} = 1 - a + \dfrac{a^2}{1+a}.$

19. The reciprocal of a number x is $\dfrac{1}{x}$. What is the product of a number and its reciprocal?

20. Give the reciprocals of each of the following: 6, m ab, $\dfrac{4}{5}$, $3\frac{1}{2}$, $a+b$, $\dfrac{1}{x}$, $\dfrac{7}{c}$, $\dfrac{1}{r+s}$, $\dfrac{a-b}{2ab}$.

21. State the rule for dividing by a fraction by using its reciprocal.

Reduce each of the following to a simple fraction in its lowest terms:

22. $(m+n)\cdot\frac{1}{m}$.

23. $\frac{a+b}{5}-\frac{5}{a+b}$.

24. $4+\frac{4}{a}+\frac{4}{a^2}$.

25. $\frac{m^2+n^2}{mn}+(m+n)$.

26. $\left(\frac{a}{b}+\frac{c}{d}\right)\div\frac{e}{f}$.

27. $\frac{a}{b}\cdot\left(\frac{c}{d}\div\frac{e}{f}\right)$.

Exercise 47

Perform the following indicated operations:

1. $a+\frac{1}{2}a$.
2. $x+(-x)$.
3. $-x+(-x)$.
4. $-x-(-x)$.
5. $\frac{m}{2}+\frac{m}{3}$.
6. $\frac{m}{3}-\frac{m}{2}$.
7. $-(a+x)^2$.
8. $(-a-x)^2$.
9. $0-(-4)$.
10. $3\cdot 0\cdot(-7)\cdot 3$.
11. $0+6$.
12. $0+(-2)$.
13. 5^2-2^2.
14. 2^3-2^2.
15. $2\cdot 3\cdot 4-2\cdot 3$.
16. $9v-9$.
17. $-(-a)^3$.
18. $-(-2)^2$.
19. 4^2+2^2.
20. -3^2+3^2.
21. $(3a\cdot 6b)\div 3$.
22. $(3a+6b)\div 3$.
23. $(4^2+6^2)\div 2$.
24. $(4^2+4^3)\div 4$.
25. $(x-y-1)\div(-1)$.
26. $(3\cdot 2^3+5\cdot 2^2-2)\div 2$.
27. $(x+y)\div(x+y)$.
28. $(x+y)\div(-x-y)$.
29. $-x^2\div(-x)^2$.
30. $a^n\div a^7$.

Reduce the following to simple fractions in their lowest terms:

31. $\frac{6\cdot 14}{3\cdot 2}$.
32. $\frac{6+14}{3+2}$.
33. $\frac{6+3x}{2+x}$.
34. $\frac{5}{5+5x}$.
35. $m\div\frac{1}{m}$.
36. $b+\frac{2}{b}$.
37. $\frac{1}{m}\div m$.
38. $2^3\cdot\frac{1}{2^5}$.
39. $\frac{2^5-2^3}{2^2}$.

40. $\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}.$

41. $m+1+\frac{1}{m}.$

42. $\frac{1}{a-b}+\frac{2b}{(a-b)^2}.$

43. $\frac{1}{p^2-q^2}-\frac{1}{(p-q)^2}.$

44. $\frac{a}{a-1}-\frac{1}{a(a-1)}.$

45. $\frac{a-\frac{1}{2}}{\frac{2}{a}}.$

46. $\left(\frac{-2^3}{(-2)^4}\div\frac{-3^5}{(-3)^5}\right)+\frac{1}{2}.$

47. $\frac{a+b}{-a-b}+\frac{x-2}{2-x}.$

48. $\frac{\frac{a}{3}+\frac{a}{4}}{\frac{a}{3}-\frac{a}{4}}.$

49. $\frac{a-\frac{1}{a}}{1+\frac{1}{a}}.$

50. $\frac{p+prt}{1+rt}-p.$

51. The total surface of a right circular cylinder is how many times the sum of its two bases?

52. The total surface of a right circular cylinder is how many times its curved surface?

53. The total surface of a right circular cylinder is how many times the total surface of a right circular cone of the same radius and whose slant height equals the altitude of the cylinder?

Express the right members of the following equations in forms more convenient for computation when finding the values of the capital letters involved:

54. $R=\frac{1}{r}-\frac{1}{r+1}.$

55. $\frac{1}{A}=a+\frac{1}{a}.$

56. $X=x+1+\frac{1}{x}.$

57. $X=\frac{2a}{2a+b}-\frac{b}{a+b}.$

58. $N=\frac{1}{n-a}-\frac{1}{n+a}.$

59. $Y=\frac{\frac{2}{3}+4a}{\frac{5}{6}+\frac{1}{2y}}.$

Exercise 48

1. Write two general numbers each of which is divisible by 7. Write their sum; also their difference. Show that the sum and the difference are each divisible by 7.

2. The area of a circle with a radius $3r$ is how many times the area of a circle whose radius is r?

Find the difference of the areas of two circles whose radii are m and n. Write this difference in the best form for computation. What is the area between two concentric circles whose radii are 40 ft. and 32 ft.?

3. $(a + 7)\ (a + 3) = ?$

4. Find the prime factors of $a - 49a^3$.

5. Find the L.C.M. of a^2bc, ab^3c^2, and ab^2c.

6. The tap will fill a certain tank in 15 minutes. The drain will empty it in 18 minutes. How long will it require to fill the tank if both tap and drain are left open?

Write a formula that gives the answer to this problem if the tap fills the tank in f minutes and the drain empties it in e minutes. Reduce this formula to a convenient form for computation.

7. The number 53 is 3 more than what number? The number 47 is 3 less than what number? Write the product $53 \cdot 47$ in the form $(a + b)\ (a - b)$ and find the product by inspection. Find the following products by the same method: $35 \cdot 45$; $65 \cdot 75$; $102 \cdot 98$; $77 \cdot 73$; $88 \cdot 82$; $107 \cdot 93$; $12\frac{1}{2} \cdot 11\frac{1}{2}$; $20\frac{1}{2} \cdot 19\frac{1}{2}$.

8. Solve the equation $r = r^2 - 12$.

9. Find all the pairs of values of x and y, both x and y being positive integers, such that $x + y = 17$.

10. How much more fencing is needed to enclose a rectangular field $(3m - 2n + 7)$ rods long and $(2m - 5n - 1)$ rods wide than is needed to enclose a field $(n - m + 6)$ rods long and $(2n - 4m - 13)$ rods wide? First indicate the operations to be performed and then perform them.

CHAPTER VI

LINEAR EQUATIONS IN ONE UNKNOWN INVOLVING FRACTIONS

41. Equations involving fractions. What is the product of $\frac{2}{3}$ by 6? Name another number by which $\frac{2}{3}$ may be multiplied to get an integral product.

By what may $\frac{5}{8}$ be multiplied to get an integral product? What is the smallest number by which $\frac{5}{8}$ may be multiplied to get an integral product?

Name a number by which you may multiply $\frac{1}{2}$ and $\frac{1}{3}$ and get integral products.

Name a multiplier of both $\frac{2}{5}$ and $\frac{3}{4}$ that will give integral products.

Multiply $\frac{5}{8}$, $\frac{5}{6}$, and $\frac{5}{12}$ by the same number so as to get an integral product in each case.

The pupil should have discovered that to get an integral product when a fraction is multiplied by an integer the multiplier must be exactly divisible by the denominator of the fraction.

A **multiple** of a given number is the product of that number by any integer. Hence a multiple of a number is exactly divisible by the number.

To get integral products when several fractions are multiplied by the same integer, the multiplier must be a common multiple of all the denominators.

Example 1. Solve for x the equation $\frac{x}{2} = 5.$

By what can $\frac{x}{2}$ *be multiplied to get an integral result? What is the result? Multiply both members of the equation by the number necessary to get an integer for the left member.*

Example 2. Solve for a the equation $\frac{2a}{3} - 1 = \frac{7}{5}.$

By what number may the two members of the equation be multiplied in order to have only integers in both members?

Multiplying both members by 15, $10a - 15 = 21.$

Adding 15 *to each member,* $10a = 36.$

Dividing each member by 10, $a = 3.6.$

Check:

$$\frac{2 \cdot 3.6}{3} - 1 = \frac{7}{5} \qquad 1.4 = \frac{7}{5}.$$

$$\frac{7.2}{3} - 1 = \frac{7}{5} \qquad 1.4 = 1.4.$$

$$\frac{4.2}{3} = \frac{7}{5}$$

Exercise 49

Solve and check the following equations:

1. $\frac{x}{5} = 3.$

2. $\frac{a}{3} + 1 = 3 - a.$

3. $\frac{a}{4} = \frac{2}{5}.$

4. $\frac{2y}{3} - 4 = \frac{5}{6}.$

5. $3m + \frac{2}{3} = \frac{m}{2} + 9.$

6. $2x - \frac{5}{2} = \frac{6x}{7} - \frac{3}{14}.$

7. $13n - \frac{9n + 8}{7} + 7 = \frac{2n + 9}{5} + 38.$

8. $\frac{1}{2} - \frac{1}{3y} = \frac{1}{4y} - \frac{5}{6y}.$

9. $\frac{1}{a + 1} + \frac{1}{a - 1} = \frac{4}{(a + 1)\ (a - 1)}.$

10. $\frac{8}{m(m - 1)} - \frac{2}{m - 1} = \frac{1}{m}.$

11. The smaller of two numbers is $\frac{2}{5}$ of the larger. Their sum is 42. What are the numbers?

12. A certain number added to both numerator and denominator of the fraction $\frac{7}{9}$ gives a fraction equal to $\frac{5}{6}$. Find the number.

13. The number of degrees on a Fahrenheit thermometer corresponding to a given number of degrees on a centigrade thermometer is given by the equation $F = 32 + \frac{9}{5}C$. Find F when C is 40. Find C when F is 0; when F is 32; when F is 23.

14. The area of a triangle is given by the equation $A = \frac{ab}{2}$, in which A is the area, a the altitude, and b the base. Find b if $A = 24$ and $a = 6$.

15. The area of a trapezoid is given by the equation $A = \frac{h(b + b')}{2}$. Find b if $A = 40$, $h = 8$, and $b' = 6$.

16. The rate at which a given interest i is produced by a given principal p in a given time t is given by the formula $r = \frac{i}{pt}$. Find i, if $r = .06$, $p = 500$, $t = 2$.

17. The horsepower of an automobile is computed by the formula H.P. $= \frac{ND^2}{2.5}$ in which N is the number of cylinders and D the diameter of the cylinder in inches. How many cylinders has a car of $31\frac{1}{4}$ horsepower whose cylinder diameter is $3\frac{1}{8}$ inches?

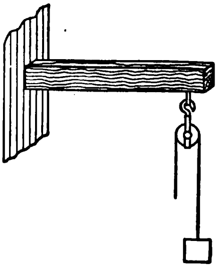

FIG. 11.

18. The beam in Figure 11 is supported at one end and supports a load at the other end. The heaviest load that a steel beam in such a position will support is given by the formula $W = \frac{6800bd^2}{L}$, in which W is the load in pounds, b the width, d the depth or thickness, and L the length of the beam, all these dimensions being in inches. What is the longest safe length for a beam to support a weight of 680 pounds if the beam is 3 inches wide, and 2 inches deep?

19. If the dividend is D, the divisor d, the quotient q, and the remainder r, then $\frac{D}{d} = q + \frac{r}{d}$. Solve this equation for r. Find r if $D = 128$, $d = 9$, $q = 14$.

20. The formula for the average of two numbers, m and n, is $a = \frac{m+n}{2}$. Find n if $a = \frac{3}{4}$ and $m = \frac{2}{3}$.

21. Solve for r the formula $m = \frac{L+r}{L-r}$.

22. Solve $w = \frac{pl}{h}$ for h, for p, and for l.

23. Solve $S = T - \frac{\sqrt{3}}{N}$ for T and for N.

24. Solve $t = \frac{2b}{3}$, for b.

25. Solve $C = \frac{\pi r d}{12}$ for r, and for d.

26. Solve $\frac{R}{10} = \frac{F}{D}$ for D.

27. Solve $H = \frac{2\pi LPN}{33{,}000}$ for L.

28. Solve $W = \frac{APb - VLg}{a}$ for V.

29. Solve $D = \frac{5280}{\pi N}$ for N.

30. Solve $b = \frac{50{,}000h}{dwn}$ for w.

Exercise 50. Review Questions

1. Divide $a^3 - b^3$ by $a - b$. State as a formula the following principle: The difference of the cubes of two numbers is equal to the difference of the numbers multiplied by the square of the first number plus their product plus the square of the second number.

2. Subtract the number whose digits in order from left to right are a, b, and c, from the number whose digits in the same order are c, b, and a.

3. Find in the shortest way possible the sum of the first 28 multiples of 6.

4. What is the square root of $n^2 - 12n + 36$?

5. Solve the equation, $2c^2 = 6c$.

6. How much greater is the sum of the edges of a rectangular solid whose edges are a, b, and c, than the sum of the three circles whose diameters are a, b, and c?

7. What is the common factor of $m^2 - 4n^2$ and $m^2 + mn - 6n^2$?

8. A man finds that he has made an automobile trip of 130 miles in 6 hours. Part of the trip was made over a paved road at the rate of 30 miles an hour and part over a poorer road at 20 miles an hour. He was delayed a half-hour by punctures. How many hours of his trip did he drive over the paved road?

9. To get 200 pounds of 4.6% milk a dairyman mixes 4% milk with 5% milk. How many pounds of each kind must he use?

10. Write with a positive sign before the fraction $-\frac{(-m-n)\ (m+n)}{(a-b)c}$.

11. Perform the indicated operations:

$$\frac{m^2 - n^2}{m} + \frac{n^2 - m^2}{n} - \frac{m - n}{1}.$$

CHAPTER VII

EXPRESSING RELATIONS BETWEEN QUANTITIES THAT CHANGE TOGETHER. GRAPHS

42. Circular pictograms. In working with numbers we are not only interested in the size of each one but we want to compare them. This comparison is more easily made if the numbers are represented graphically, by means of lines, curves, or other geometric figures. We shall now consider certain uses made of graphs, in applied problems.

EXAMPLE 1. In 1910 there were in the United States 5,516,163 illiterates 10 years of age and over who were divided into the following classes:

NATIVE WHITE		Foreign Born White	Negro	All Others
Native Parentage	Foreign or Mixed Parentage			
1,378,884	155,388	1,650,361	2,227,731	103,799

The comparative sizes of these groups can be best seen if the total illiterate population is represented by a geometric figure of which each group is a part. A good figure for this purpose is a circle. In constructing Figure 12 the per cent that each group is of the whole was first computed. For example, the group of native parentage is 25.0% of the whole. Then they should be represented by 25.0% of the *circle, or* 90°. The number of degrees that should represent each of the other groups was then computed and the circle properly divided by using a protractor.

Figure 12 is called a **circular pictogram.** It is much used in showing the relations of the parts and the whole.

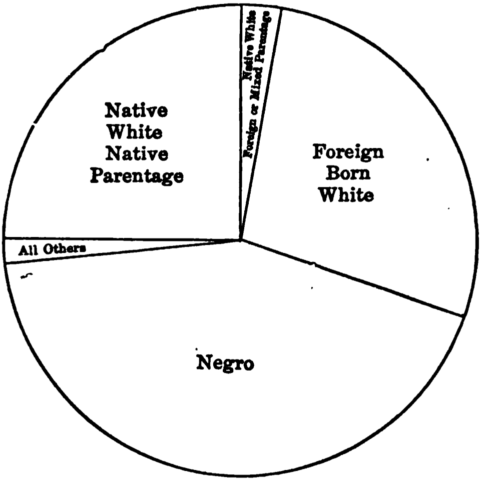

FIG. 12.

Exercise 51

1. The population of the United States in 1910 was composed of the following groups:

NATIVE WHITE		Foreign Born White	Negro	All Others	Total
Native Parentage	Foreign or Mixed Parentage				
49,488,575	18,897,837	13,345,545	9,827,763	412,546	91,972,266

Represent these numbers in a circular pictogram. Compare this pictogram with Figure 12. Which group has the greatest per cent of illiterates? Which has the least? Can you arrange these groups according to their percentages of illiteracy simply by comparing the graphs?

2. The land areas of the earth's surface expressed in square miles are as follows:

Africa,	11,622,619	Australia,	3,312,613
North America,	8,589,257	Europe,	3,872,561
South America,	7,570,015	Polar Regions,	5,081,935
Asia,	17,306,000		

Represent these areas in a circular pictogram.

3. The average annual number of clear, partly cloudy, and cloudy days in Chicago from 1873 to 1910 is as follows: clear, 115.6; partly cloudy, 135.6; cloudy, 113.9. Illustrate with a pictogram. The illustration may be improved by proper shading.

4. The following are suggested by students of household economy as proper distributions of incomes of various sizes in families of four or five persons:

Income	Food	Rent	Operating Expenses	Clothes	Other Expenses and Savings
$500 to $800	45 %	15 %	10 %	10 %	20 %
$800 to $1000	30 %	20 %	10 %	15 %	25 %
$1000 to $2000	25 %	20 %	15 %	20 %	20 %
$2000 to $4000	25 %	20 %	15 %	20 %	20 %

Represent the per cents for each of the four groups of salaries by a circular pictogram.

5. The following table gives the range of salaries received by school teachers in Illinois in a recent year, and the number of teachers receiving different salaries:

Salary	No. of teachers receiving	Salary	No. of teachers receiving
Less than $600	7,376	From $1,800 to $2,499	5,441
From $600 to $999	13,422	From 2,500 to 2,999	458
From 1,000 to 1,399	5,977	Over 3,000	878
From 1,400 to 1,799	3,047		

Represent these numbers in a circular pictogram.

43. Bar graphs used in comparing statistics. Bar graphs are much used in the treatment of statistics. We shall now have some examples of their use in some typical problems.

Example. It was desired to compare the grades in English and mathematics of a group of students. In the school in which these grades were made 70 is a passing mark, 65 is a condition, and below 65 is failing.

The following grades of 39 students were chosen.

Student,	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
Grade in English,	65	92	90	82	80	65	60	85	80	65	75	65	85
Grade in mathematics,	80	95	95	91	83	65	65	82	94	77	92	82	88

Student,	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**
Grade in English,	60	63	90	75	80	77	75	84	80	84	89	65	80
Grade in mathematics,	79	80	92	87	83	73	86	89	86	89	86	91	65

Student,	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**
Grade in English,	73	60	85	75	60	79	88	75	65	87	72	83	80
Grade in mathematics,	65	65	82	72	60	74	95	76	65	90	65	93	83

The table shows grades scattered over a wide range. With the grades arranged as in the table, it is difficult to compare them. We then look for a better arrangement. Since we shall want to know the highest and lowest grades, and the number of students getting each different grade, we arrange the grades as in the table below. This table shows the frequency of each grade, that is, how many students get each grade, and is called **a frequency table.** For example, the table shows that one student got the grade 83 in English, and three students got that grade in mathematics.

Grade,	60	63	65	70	71	72	73	74	75	76
Number getting this grade in English,	4	1	6	0	0	1	1	0	5	0

Number getting this grade in mathematics,	1	0	7	0	0	1	1	1	0	1

Grade,	77	78	79	80	81	82	83	84	85	86
Number getting this grade in English,	1	0	1	6	0	0	1	2	3	0
Number getting this grade in mathematics,	1	0	1	2	0	3	3	0	0	3

Grade,	87	88	89	90	91	92	93	94	95
Number getting this grade in English,	1	1	1	2	0	2	0	0	0
Number getting this grade in mathematics,	1	1	2	1	2	2	1	1	3

In this table we have made a class for each grade, that is, a class, or group, for 70, one for 71, and so on. The range or **class interval** covered by each class is then one. With this small number of grades to examine we have made too many classes, that is, we have made the class interval too small. We should then make the class interval larger so that the grades will not be scattered over so many intervals. Let us make the class interval five. We then have the following frequency table:

Grade,	60–64	65–69	70–74	75–79
Number making these grades in English,	5	6	2	7
Number making these grades in mathematics,	1	7	3	3

Grade,	80–84	85–89	90–94	95–100
Number making these grades in English,	9	6	4	0
Number making these grades in mathematics,	8	7	7	3

This table gives us a much better notion of how these grades have been distributed. The comparison will be more easily made if graphs are made as in Figure 13. The graph in solid lines represents the grades in mathematics.

From the graph, Figure 13, tell how many pupils failed in each subject; how many were conditioned in each. In what class interval do most grades occur in mathematics? In English? How many grades, of 80 or above, were given in each subject? In which subject do you think there is the better chance of getting a grade of 80 or above?

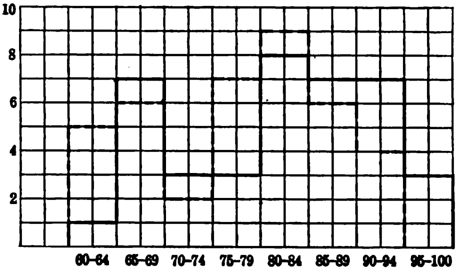

FIG. 13.

Find the average grade in each subject.

Exercise 52

1. The following are the averages of all grades in science and of all grades in manual arts made by each of 31 students who graduated from a certain school. The students are arranged alphabetically.

Student,	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
Science,	87	79	88	86	74	79	74	75	76	84	76
Manual Arts,	82	79	88	85	80	85	79	84	86	85	74

Student,	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**
Science,	78	80	92	88	82	83	85	79	83	82
Manual Arts,	83	78	84	80	86	82	84	75	79	87

Student,	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**
Science,	79	79	89	80	83	83	88	86	78	78
Manual Arts,	84	88	85	84	81	84	70	89	85	82

Arrange the grades in each subject in a frequency table. Construct a bar graph for each table. Use care in choosing the class interval in making the frequency polygons. Find the average in each subject. In which subject have more grades of 80 or above been made?

2. The average of all of the grades in manual arts and of all of the grades in English was found for each of 508 graduates of a certain school. The results are given in the following frequency tables:

Grade,	**64**	**67**	**68**	**69**	**70**	**71**	**72**	**73**	**74**	**75**
Number making grade in English,	1	4	1	2	6	8	17	16	25	19
Number making grade in manual arts,	0	0	0	0	2	2	4	4	4	5

Grade,	**76**	**77**	**78**	**79**	**80**	**81**	**82**	**83**	**84**	**85**
Number making grade in English,	27	30	31	37	24	29	32	31	30	35
Number making grade in manual arts,	5	12	21	36	41	47	55	53	61	48

Grade,	**86**	**87**	**88**	**89**	**90**	**91**	**92**	**93**	**97**
Number making grade in English,	21	19	20	17	11	7	3	4	1
Number making grade in manual arts,	42	27	20	11	3	2	3	0	0

Make a bar graph for each table. Find the average of the grades in each subject.

3. The results of giving the army intelligence tests to 7000 drafted men at Camp Hancock are given in the following table. The table shows the per cent of the group that made each rank. The table also shows what is called

Rank		Per Cent Drafted Men Making	Per Cent in Normal Distribution Making
A	(Very superior)	.6 %	5 %
B	(Superior)	2.7 %	15 %
C +	(High average)	6.5 %	15 %
C	(Average)	19.0 %	30 %
C –	(Low average)	24.0 %	15 %
D	(Inferior)	35.0 %	15 %
D –	(Very inferior)	12.2 %	5 %

a "normal distribution." For example, in a normal distribution, 5% would make *A*; in the test at Camp Hancock, only .6% made *A*.

Construct a bar graph for the distribution at Camp Hancock and for the normal distribution.

4. The following table gives the enrollment in 59 rural schools in a county in Illinois in a certain year.

Number of students enrolled,	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Number of schools having the enrollment,	1	1	2	1	1	5	0	2	2	3

Number of students enrolled,	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
Number of schools having the enrollment,	1	7	3	1	3	2	4	1	1	2

Number of students enrolled,	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**
Number of schools having the enrollment,	1	3	1	1	2	1	1	0	1

Number of students enrolled,	**40**	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**50**
Number of schools having the enrollment,	0	0	1	0	0	1	0	2	1

The distribution is scattered over too wide a range for an examination to tell much about it. It will be better to make the class interval larger, say five. Then the groups will be from 11 to 15, 16 to 20, and so on. After making this grouping construct a graph. Find the average enrollment.

5. According to the United States Life Tables of 1910, of 100,000 persons born alive the following table gives the number dying in each decade:

Decade,	0 yr. to 10 yr.	10 yr. to 20 yr.	20 yr. to 30 yr.	30 yr. to 40 yr.
Number dying in this decade,	17,542	2,384	4,295	5,737

Decade,	40 yr. to 50 yr.	50 yr. to 60 yr.	60 yr. to 70 yr.	70 yr to 80 yr.	80 yr. to 90 yr.	90 yr. to 100 yr.	100 yr. to 107 yr.
Number dying in this decade,	7,582	11,322	17,322	20,104	11,844	1,828	40

Construct a graph for this table. Do you know any cause for the large number of deaths in the first decade? In which decade is the number of deaths least? Greatest?

44. Quantities that change together. The student has had much experience with quantities that are so related that a change in one is accompanied by a change in the other. He has seen the temperature during the day change with the time; he knows that the cost of a basket of eggs at 30¢ a dozen depends upon the number of dozens; he knows that the area of a square changes with the length of its side. The student's attention has been called many times in this book to the relations between quantities that change together. The object of this chapter is to find and use ways of expressing such relations.

Exercise 53

Tell what each of the following depends upon:

1. The cost of a farm at $250 an acre.
2. The area of a rectangle with a given base.
3. The amount of interest on $1000 for 2 years.
4. The size of the product if the multiplier is 300.
5. The size of the quotient if 300 is the dividend.
6. The number of oranges that can be bought for 50¢.
7. The height of the mercury in the thermometer.
8. The amount that a boy makes who is selling aluminum ware at 40% commission.
9. The area of a circle.

Name one or more things upon which each of the following depends:

10. The amount of interest that a man pays on borrowed money.

11. Your grade in mathematics.

12. The distance a batted ball goes.

13. The cost of a barrel of flour.

14. The height that a rubber ball will bounce.

15. The volume of a rectangular solid.

16. A boy allows himself a certain amount of money to spend each month at the movies. What will be the effect upon his attendance if the price of movies goes up?

17. Another boy allows himself to go to the movies a certain number of times each month. How will the increased price affect him?

18. Name two quantities that change together so that if one increases the other increases.

19. Name two quantities that change together so that if one increases the other decreases.

20. Name two quantities such that if one decreases the other decreases.

21. Can you name two quantities that vary together so that if one increases the other increases a while and then decreases?

22. The amount, A, of $1 at 6% simple interest at the end of n years is given by the formula $A = 1 + .06n$. Find A for values of n from 1 to 10 inclusive, and complete this table.

n	1	2	3	4	5	6	7	8	9	10
A										

Does A increase or decrease as n increases?

23. Does the light received from an electric lamp become more or less intense as you move away from it? The formula $I = \frac{100}{d^2}$ gives the number of candlepower, I, received from a light at a distance d feet from it. Compute I, correct to .1, for integral values of d from 1 to 10 inclusive, and make a table as in the preceding problem. Does I increase or decrease as d increases? What kind of value will I have when d becomes very large?

24. The formula $h = 192t - 16t^2$ gives the height, h, in feet at the end of t seconds of a body thrown vertically upward at the rate of 192 ft. per second. Compute values of h for integral values of t from 0 to 12 inclusive. Does h increase or decrease as t increases from 0 to 12?

25. Are the quantities A and n in exercise 22, two such quantities as are asked for in either of the exercises 18, 19, 20, or 21? Answer the same question for the quantities I and d in exercise 23; also for h and t in exercise 24.

45. Ways of expressing relations between quantities that change together. The student has studied tables that show relations between quantities, such as tables giving the rainfall or temperature for a given time, and interest tables. He has also made graphs from these tables. He has also studied many formulas that express relations between quantities that change together. Thus he has seen three important ways of expressing such relations:

By tables.
By graphs.
By formulas.

Since intelligent beings began to observe and compare natural events they have been interested in trying to determine why things happen just so, that is, what are the causes of things, and how one event is related to another. They have tried to find out just how and why the heavenly bodies move, what causes the change of seasons, how bodies fall, how plants and animals develop, what connection there is between mosquitoes and malaria, whether it is unlucky to sit 13 at the table, and whether potatoes should be planted in a particular " sign " of the moon. Wise persons have seen that in order to find out just how and why an event occurs it is not sufficient to observe it just once, but many times. Furthermore, they have seen that it is not safe to depend upon the memories of people, but that careful records must be kept if we are to be sure just how things happened. If we observe for a long time and keep a careful record, it may appear that Friday is not more unlucky than other days. So a great many observations have been recorded about the weather, the growth of populations, prices of farm products, the amount of fuel consumed by certain types of steam engines, — in fact about all sorts of things that men are making inquiries about. When these observations are recorded in numerical form they are called **statistics.**

Year	Population	Year	Population
1790	3,929,214	1860	31,443,321
1800	5,308,483	1870	38,558,371
1810	7,239,881	1880	50,155,783
1820	9,638,453	1890	62,947,714
1830	12,860,702	1900	75,994,575
1840	17,063,363	1910	91,972,266
1850	23,191,876	1920	105,683,108

We can go to these tables of statistics and get the facts that are recorded. For example, we can go to the report of the census of the United States and find the population for each decade from 1790 to 1920. If we arrange these facts in order as in this table, we can see with some clearness how fast the population has increased. But if a graph is made, as in Figure 14, the rate of increase

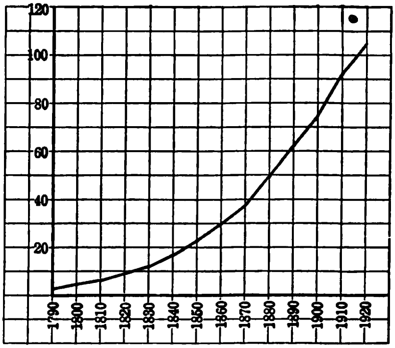

Fig. 14.

is shown much more clearly. The graph gives an impression at a glance that it takes much examination of the table to discover. One unit on the vertical axis represents 10,000,000 people.

It has not been possible to find exact mathematical laws, expressible in formulas, that will describe exactly the facts found in most tables of statistics. Large amounts of statistical materials have been collected concerning the weather, but the weather bureau has not yet been able to write down formulas from which we may compute what the rainfall will be next summer or what the temperature will be on the Fourth of July. Departments of agriculture give us statistics of crop reports and of prices of farm products from day to day, month to month, and year to year. But no one has yet been able to write a formula from which we can compute what the price of hogs will be next June, although such a formula would be very useful and would probably stimulate farmers' sons in their study of algebra. But many useful conclusions can be drawn from the study of these tables of statistics. By examining the weather reports for a long period we can draw fairly safe conclusions as to when the first killing frost will come in the fall, what the rainfall is likely to be each month, and what the average temperature is likely to be through the year. From an examination of market reports we may find how the prices of hogs are likely to vary from month to month; also the comparative prices of hogs and corn, from which the farmer may decide whether it was more profitable to feed his corn or to sell it.

This examination of tables of statistics is made much easier if the statistics are first represented graphically. This is especially true if we are making comparisons of different tables of statistics.

46. Coördinate axes. In making graphs such as that in Figure 14 we make use of two fixed perpendicular lines of reference. Such lines are usually denoted by XX' and YY'

(See Figure 15) and are called **coördinate axes** and their intersection the **origin.**

The horizontal line XX' is called the X-axis and the vertical line YY' is called the Y-axis. The horizontal distance from the Y-axis to a point P is called the **abscissa** or x-value of the point. The vertical distance from the X-axis to P is called the **ordinate** or y-value of the point. The abscissa and the ordinate of a point are together called the **coördinates** of the point. It is the custom to take distances measured to the right from YY' as positive and those to the left as negative; those measured upward from XX' as positive and those measured downward as negative.

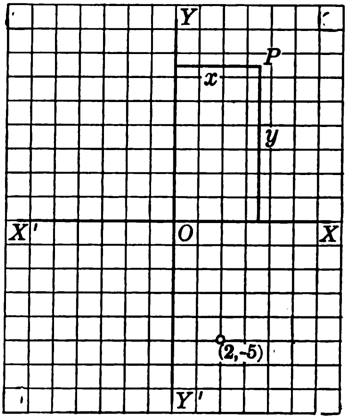

FIG. 15.

47. Plotting of points. If we have given any two numbers, say 2 and -5, we can find one and only one point that has the first number for its abscissa and the second number for its ordinate. To find the location of the point for which $x = 2$ and $y = -5$, we start at the origin and measure 2 units to the right along the X-axis, and from this point we measure downward a distance 5. The point thus reached may be represented by the symbol $(2,-5)$. The symbol (a,b) denotes a point whose abscissa is a and whose ordinate is b. When a point is located in the manner described above it is said to be **plotted.**

48. Use of coördinate paper. In plotting points and in obtaining the geometric pictures we are about to make, it is convenient to use coördinate paper. This is paper ruled both horizontally and vertically as shown in Figure 15.

Exercise 54

1. Plot the points (4,5), (4,–5), (–4,5), (–4,–5).
2. Plot the points (0,6), (0,0), (–2,0), (0,–6), (3,0).
3. Where are all the points in the plane where $x = 2$?
4. Where are all the points in the plane where $y = -6$?
5. Where are all the points in the plane where $x = 0$?
6. Where are all the points in the plane where $y = 0$?
7. If a point moves parallel to the x-axis, which of its coördinates remains constant?
8. If a point moves parallel to the y-axis, which of its coördinates remains constant?
9. Draw a triangle with its vertices at the points (0,3), (–2,–4) and (1,3).
10. Draw a quadrilateral with its vertices at the points (–1,3), (–2,–4), (3,–1), and (5,4).
11. Plot the points whose coördinates are these pairs of values of x and y:

x	–10	–8	–6	–4	–2	0	2	4	6	8	10
y	–7	–5	–3	–1	1	3	5	7	9	11	13

How do these points lie with respect to each other?

12. In Figure 14 the time is measured along which axis? The population is measured along which axis? How many years are represented by one unit? How many millions of population are represented by one unit? Can you name some things to be considered in deciding what scale is to be used in making a graph?
13. From the equation $y = 3x - 2$ find values of y for integral values of x from –4 to 10 inclusive, and complete this table.

x	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
y															

Plot the points whose coördinates are these pairs of values of x and y. How do these points lie with respect to each other? As x ncreases does y increase or decrease?

14. From the equation $y^2 = 4x$ find the two values of y corresponding to each of the integral values of x from 0 to 9 inclusive, and complete the table. For example when $x = 1$, $y^2 = 4$, and $y = +2$, or -2. Find square roots to the nearest .1.

x	0	1	2	3	4	5	6	7	8	9
y	0	2	2.8							
y	0	-2	-2.8							

Plot all the points whose coördinates are given by taking each value of x with each of the two corresponding values of y, for example (1, 2), (1, −2), (2, 2.8), (2, −2.8), and so on. As x increases what does y do?

5. Plot the points given by using the table in exercise 22, page 91. Use values of n as abscissas and values of A as ordinates.

49. Using axes in making graphs. We shall give an example of using coördinates and coördinate axes in making graphs of some statistics.

The Bureau of Labor Statistics of the United States Department of Labor publishes tables which show the average wholesale prices of the most important commodities from year to year for the last 30 years. These tables are important as showing the change in the cost of living, about which there is always much discussion. The following table shows the comparative average wholesale prices of all the commodities considered by the Bureau from 1900 to 1919. The average wholesale price in 1913 is taken as a standard to which other prices are compared, and is called 100. Thus, the average wholesale price of all commodities in 1905 is 95% of that in 1913, and that of 1919 is 212% of that in 1913.

Year,	**1900**	**1901**	**1902**	**1903**	**1904**	**1905**	**1906**
Comparative price,	80	79	85	85	86	85	88
Year,	**1907**	**1908**	**1909**	**1910**	**1911**	**1912**	**1913**
Comparative price,	94	91	97	99	95	101	100
Year,		**1914**	**1915**	**1916**	**1917**	**1918**	**1919**
Comparative price,		100	101	124	176	196	212

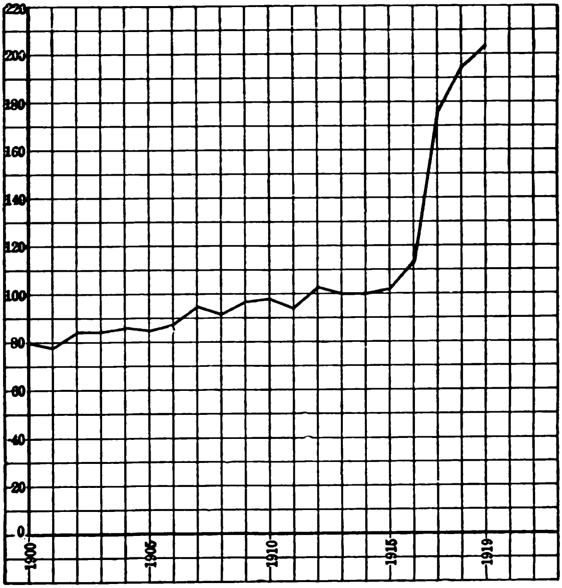

FIG. 16.

To represent these prices by a graph we lay off the years along the horizontal axis and the prices along the vertical axis. One horizontal space represents one year and one vertical space represents 10 price units. We first find the point that represents the price for each year and then connect adjacent points by straight lines.

In what year were prices highest? Lowest? Where is the graph steepest? What does that indicate in the change of prices? Is the graph anywhere horizontal? If so, what does that indicate? Was there a tendency for prices to rise or fall from 1900 to 1913? Why did prices rise so rapidly after 1915?

Exercise 55

Make graphs of the following statistics:

1. The following are the highest prices paid for beef steers each year from 1900 to 1920 in the Chicago market as compiled by the *Chicago Drovers Journal.*

Year	Price	Year	Price	Year	Price	Year	Price
1900	$7.50	1905	$7.00	1910	$8.85	1915	$11.60
1901	8.00	1906	7.90	1911	9.35	1916	12.60
1902	9.00	1907	7.60	1912	11.25	1917	17.90
1903	6.85	1908	8.40	1913	10.25	1918	20.50
1904	7.65	1909	9.50	1914	11.40	1919	21.50
						1920	19.25

The highest price in this list is the highest price on record. In what year was it? Do you know any cause of the rise of prices from 1914? When did the World War end? Did the prices for beef steers decline then? In what year was the average lowest? Do you know any cause for the low price in that year? What has been the general trend of prices since 1900?

2. In this table are given the average prices received by farmers for eggs on the first of each month in the years 1910, 1911, 1912, 1917, 1918, 1919.

	Jan.	Feb.	Mch.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
1910	30.5	28.9	22.9	18.6	18.6	18.3	18.2	17.6	19.4	22.4	25.3	29.0
1911	30.4	22.1	16.5	14.9	14.7	14.5	14.2	15.5	17.4	20.0	23.5	28.7
1912	29.5	29.1	24.5	17.8	17.1	16.7	16.7	17.4	19.1	22.0	25.9	29.7
1917	37.7	35.8	33.8	25.9	30.0	31.1	28.3	29.8	33.2	37.4	39.4	43.3
1918	46.3	49.4	40.4	31.2	31.0	29.8	30.7	34.4	36.4	41.6	47.2	55.0
1919	57.2	48.3	33.1	34.3	36.8	38.6	36.8	39.3	41.0	44.7	54.0	61.9

Make a graph of the prices of each year, all six of the graphs on the same sheet. In which year were the prices highest? Lowest? What is the highest price quoted? When? At what time of year are the prices usually highest? Lowest? Do you know any causes for prices rising from 1917 to 1919?

3. In this table are given first, the average prices in cents per bushel, paid farmers in the United States for corn on the first of each month in the years 1910, 1911, 1918, and 1919; second, the average prices in dollars per hundred pounds paid farmers for hogs on the fifteenth of each month in the same years. These figures are taken from the Yearbook of the Department of Agriculture for 1919.

	1910		1911		1918		1919	
	Corn	Hogs	Corn	Hogs	Corn	Hogs	Corn	Hogs
Jan. . . .	62.3	7.76	48.2	7.44	134.8	15.26	144.7	15.69
Feb. . . .	65.2	7.87	49.0	7.04	138.8	15.03	138.1	15.53
Mar. . . .	65.9	8.93	48.9	6.74	154.3	15.58	137.2	16.13
Apr. . . .	65.5	9.26	49.7	6.17	153.6	15.76	149.6	17.39
May . . .	63.5	8.59	51.8	5.72	155.7	15.84	162.6	18.00
June . . .	65.2	8.46	55.1	5.66	152.5	15.37	171.2	17.80
July . . .	66.2	8.15	60.0	5.92	153.7	15.58	176.5	19.22
Aug. . . .	67.2	7.78	65.8	6.54	159.7	16.89	191.2	19.30
Sept. . . .	66.3	8.27	65.9	6.53	165.7	17.50	185.4	15.81
Oct. . . .	61.1	8.08	65.7	6.09	159.5	16.50	153.9	13.88
Nov. . . .	52.6	7.61	64.7	5.86	140.3	15.92	133.4	13.36
Dec. . . .	48.0	7.16	61.8	5.72	136.5	15.82	134.9	12.66

It will be interesting to see how the prices rise and fall from month to month in each year; how the prices in the different years compare; and to see if the price of hogs rises and falls with the price of corn. This can be best accomplished if all the graphs of corn prices are put on one sheet and those for the prices of hogs on another. The student might use a solid line for 1910, a broken line for 1911, a broken line with dots, —·—·—·, for 1918, and a dotted line, · · · · for 1919. Are there any times of the year when prices of corn or hogs appear to be highest? Lowest? Which rose more rapidly, prices of corn or of hogs? Do the prices of hogs appear to rise and fall with the prices of corn? Find the average price of corn and of hogs in 1910 and 1919. What was the per cent of increase of each?

4. The trend of prices paid to farmers for important crops is indicated in the following figures. The average price paid December 1 in the 43 years from 1866 to 1908 for wheat, corn, oats, rye, buckwheat, potatoes, hay, flax, and cotton is taken as 100. The average prices of these articles on this basis are given in the table for the years 1910–1919. For example, the average price for these articles in 1910 was 129.5% of the average in the 43 years.

Year,	1910	1911	1912	1913	1914
Average,	129.5	132.8	132.8	128.1	132.4

Year,	1915	1916	1917	1918	1919
Average,	132.1	162.1	259.5	277.7	290.0

Make a graph of these averages.

5. The following figures are given by the United States Department of Agriculture as the average wages with board paid for male farm labor. The wages are given in dollars per month.

Year,	1910	1911	1912	1913	1914
Wages,	19.21	20.18	20.81	21.38	21.05

Year,	1915	1916	1917	1918	1919
Wages,	21.26	23.25	28.87	34.92	39.82

Make a graph of these averages. Compare this graph with the one made for the preceding exercise. Which rose more rapidly, wages of farm labor, or the selling prices of the leading farm crops?

6. From the following table we can compare the increase in teachers' salaries in Illinois with the increase in the cost of living from 1913 to 1920. The average salary and the cost of living in 1913 are taken as the basis and are each counted as 100.

Year,	1913	1914	1915	1916	1917	1918	1919	1920
Average salary,	100	103	106	113	116	122	128	163
Cost of living,	100	100	101	124	176	196	212	249

A graph of these figures will show very vividly how teachers' salaries have lagged behind the cost of living. (The figures on the cost of living are from a report of the U. S. Department of Labor. The figures for 1920 do not include December.)

7. The following table gives the number of hours from sunrise to sunset on the twentieth of each month in Chicago, Illinois.

MONTH . . .	JAN.	FEB.	MAR.	APR.	MAY	JUNE	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.
Possible hours of sunshine .	9.6	10.8	12.2	13.6	14.7	15.2	14.8	13.7	12.3	10.9	9.6	9.1

Make a graph of these figures.

8. This graph shows the variation in the acreage, yield per acre, and the total production of the potato crop in the United States from 1909 to 1917. The average for this period is taken as 100, and the returns for each year are compared with this average. For example, in 1909, the acreage was about 98% of the average, the yield per acre about 109% of the average, and the total production about 107% of the average. Does the graph of the total production follow more nearly the graph of the acreage or the graph of the yield per acre? Is there any year in which a marked change in the acreage probably made a marked change in the total production?

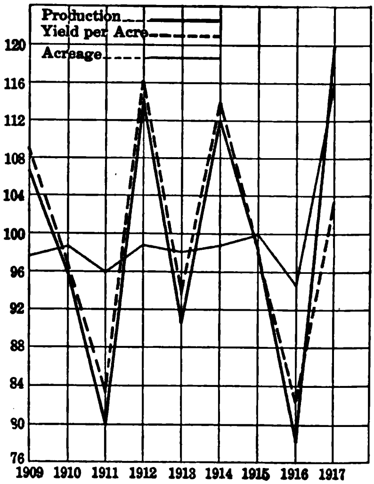

FIG. 17.

The following table gives the average farm price in cents per bushel for potatoes from 1909 to 1917.

Year,	1909	1910	1911	1912	1913	1914	1915	1916	1917
Price,	70.8	56.4	80.6	72.5	64.3	64.4	54.4	114.1	164.9

Find the average price per bushel for this period. Find what per cent each year's price is of this average. These per cents should be plotted on the same scale as the graph in the text. The student can then easily see whether the price of potatoes changed with the total production. Does "law of supply and demand" seem to apply here?

9. The following table gives the average height calculated from the data of 45,151 boys and 43,298 girls in the cities of Boston, St. Louis, Milwaukee, Worcester, Toronto, and Oakland, Cal., from the approximate ages of 5.5 years to 18.5 years.

Av. Age . .	5.5	6.5	7.5	8.5	9.5	10.5	11.5	12.5	13.5	14.5	15.5	16.5	17.5	18.5
Boys' av. ht. in inches .	41.7	43.9	46.0	48.8	50.0	51.0	53.6	55.4	57.5	60.0	62.9	64.9	66.5	67.4
Girls' av. ht. in inches .	41.3	43.3	45.7	47.7	49.7	51.7	53.8	56.1	58.5	60.4	61.6	62.2	62.7	...

Make graphs of these data. Do the graphs give any interesting information?

10. The following table gives the average weight calculated from the data of about 68,000 children in the cities of Boston, St. Louis, and Milwaukee, from the approximate age of 6.5 to 16.5 years.

Av. Age . .	6.5	7.5	8.5	9.5	10.5	11.5	12.5	13.5	14.5	15.5	16.5
Boys' av. wt. in pounds .	45.2	49.5	54.5	59.6	65.4	70.7	76.9	84.8	95.2	107.4	121.0
Girls' av. wt. in pounds .	43.4	47.7	52.5	57.4	62.9	69.5	78.7	88.7	98.3	106.7	112.3

Make graphs of these data. At what ages are girls heavier than boys? Taller?

50. Variables and constants. Let t represent the time since you started from home to school and d represent the distance that you are from home. As you proceed both t and d change, and take on many different values. Then t and d are called variables.

Definition. *A number which may take on different values in the same discussion is called a* **variable.**

If you throw a stone into a pool of water a circular wave starts which increases in diameter. If r represents the radius of this wave and c represents its circumference, then $c = 2\pi r$. In this case c and r are variables. But π has only one value and is called a constant.

Definition. *A number that has the same value throughout a discussion is called a* **constant.**

In the equation $c = 2\pi r$, for every value given r we can find a value of c, and if r changes c changes. Then c is called a function of r.

Definition. y is called a **function** of x if, when x is given, y is determined.

For example, $y = 2x + 6$, and $y = x^2$, are examples of y expressed as a function of x.

In each of the first 17 problems in Exercise 53 one or more variables appear. Find them in each case.

In the problems in Exercise 55 the variables change together. In Problem 1 the price changes with the time. In Problem 9 the height and weight change with the age of the child. Height and weight are then functions of the age. In each of the problems in this exercise name a variable that is a function of another.

The notions of variable and function are very common in everyday life and also very important. Prices, population, rainfall, distances passed over by moving bodies, interest accumulating on a debt, the amount of coal consumed by an engine, the work done by a motor, are examples of variables that are functions of other variables. Interest, for example, is a function of the time, the rate, and the principal.

Exercise 56

1. A boy placed $100 in a bank which pays 4% interest. How much interest was due him in n years? As the

number of years increased what happened to the amount of interest due? What are the variable quantities in this problem? What are the constants?

2. If the base of a rectangle is increased while the altitude remains constant what happens to the area? What are the variables and the constant in this problem?

3. If the base of a triangle is increased while the area is kept constant what happens to the altitude? What are the variables and the constant?

4. If the number of articles bought at a fixed price is increased what happens to the total cost? What are the variables and the constant?

5. A boy throws a board into a running stream. What quantities are variables? Which are constants?

6. A boy blows up a rubber balloon. What happens to the amount of air in the balloon? To the radius? The volume? What other variables and constants are involved?

51. Advantages of expressing relations between variables by formulas. We get exact knowledge of what is going on in the world about us by counting, weighing, and measuring, recording the results, and then trying to find out relations between these results. These results are sometimes set down in tables as in the examples in Exercise 55. By examining these tables and the graphs made from them we are often able to find out many useful relations. The student has made such examinations in this and the preceding chapter. Many times these relations can be expressed in formulas, as in the case of falling bodies, where $s = \frac{1}{2}gt^2$ gives us the number of feet, s, that a body falls from rest in t seconds. In physics it has been found that when a gas is put under pressure the product of the volume and the pressure is always the same. This fact may be written $pv = k$, where p represents the pressure, v the volume, and k is a constant. The student has had many other similar formulas which express relations between variables. A great advance has been made in the solution of a problem when the solution can be expressed in terms of a formula. The principal object of applied mathematics is to express relations between variables by formulas.

A formula enables us to predict what will happen under given circumstances. We can predict what the compound interest on \$500 will be in 7 years at 6%, or how far a body will fall in 10 seconds, or what the horsepower of a certain type of gasoline engine will be, because we have formulas that give the answers to such questions. But we cannot tell what the weather will be next New Year's Day, or what the population of the United States will be in 1940, because those problems have not been solved so completely as to admit an answer stated as a formula.

By solving a formula for the different letters involved we are able to solve different types of problems. Tell the kind of problem that can be solved by each of these formulas:

$$p = rb;\ b = \frac{p}{r};\ r = \frac{p}{b}.$$

The formula is the most satisfactory method of expressing the relation between variables, or expressing one variable as a function of one or more other variables. We shall examine certain types of such functions and find their graphs.

Example 1. The parcel post rate depends upon the weight of the parcel and the distance it is sent. For places in the third zone, that is, places whose distance is 150 to 300 miles, the rate in cents is 4 more than twice the number of pounds. This may be expressed by the formula, $r = 2p + 4$. For the third zone the parcel must not weigh more than 50 pounds. Hence we have r expressed as a function of p where p may take integral values from 1 to 50.

To make the graph for this equation we will let one space along the horizontal axis represent 5 pounds and one space along the vertical axis represent 5 cents. We then compute the postage for multiples of 5 pounds up to 50 pounds. These results are given in the table.

p	5	10	15	20	25	30	35	40	45	50
r	14	24	34	44	54	64	74	84	94	104

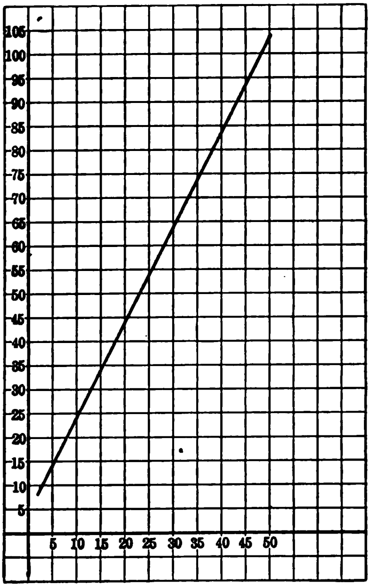

FIG. 18.

We now find the points that represent the postage for each weight given in the table. These points are now connected and are seen to lie on a straight line.

From the graph find the postage on a 10 pound parcel; on a 15 pound parcel. Find the weight of a parcel whose postage is 50 cents; 30 cents; 34 cents; 22 cents.

Exercise 57

1. For places in the fifth zone, that is, places from 600 to 1000 miles distant, the parcel post rate in cents is 2 more

than six times the number of pounds. Express this rule as a formula and find its graph.

2. If i represents the interest on $100 for n years at 6%, then $i = 6n$. Make a graph of this formula for values of n from 1 to 10. Which variable is expressed as a function of the other here?

3. If F represents the reading of a Fahrenheit thermometer and C that of a centigrade thermometer, then $F = \frac{9}{5}C + 32$. Here F is expressed as a function of C. Make a graph of this formula for values of F from 0° to 70°. From the graph find F when $C = 30°$; find C when $F = 32°$; when $F = 60°$.

4. Make a graph of the equation $y = 2x - 7$, for values of x from -4 to 10.

5. Make a graph of the equation $x + y = 6$, for values of x from -8 to 8.

52. Graphs of equations. The graph in each equation in Exercise 57 is a straight line. The five equations used may be written

(a) $r - 2p = 4$.
(b) $i - 6n = 0$.
(c) $F - \frac{9}{5}C = 32$.
(d) $y - 2x = -7$.
(e) $x + y = 6$.

Each formula involves two letters. Only the first power of each letter appears. Then each of these equations is in the general form

$$ax + by = c.$$

In (a) $a = 1$, $b = -2$, and $c = 4$. What are a, b, and c in each of the equations (b), (c), (d), and (e)?

Let us take another equation of this form,

$$2x + 3y = 12.$$

What are a, b, and c here? We wish now to find the graph of this equation. We can find corresponding values of x and y more easily if we solve the equation for y in terms of x.

Transposing, $3y = 12 - 2x$.
Dividing by 3, $y = 4 - \frac{2}{3}x$.

By giving x all integral values from -10 to $+10$ we get the following corresponding values of x and y.

x	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
y	$10\frac{2}{3}$	10	$9\frac{1}{3}$	$8\frac{2}{3}$	8	$7\frac{1}{3}$	$6\frac{2}{3}$	6	$5\frac{1}{3}$	$4\frac{2}{3}$	4	$3\frac{1}{3}$	$2\frac{2}{3}$	2	$1\frac{1}{3}$	$\frac{2}{3}$	0	$-\frac{2}{3}$	$-1\frac{1}{3}$	-2	$-2\frac{2}{3}$

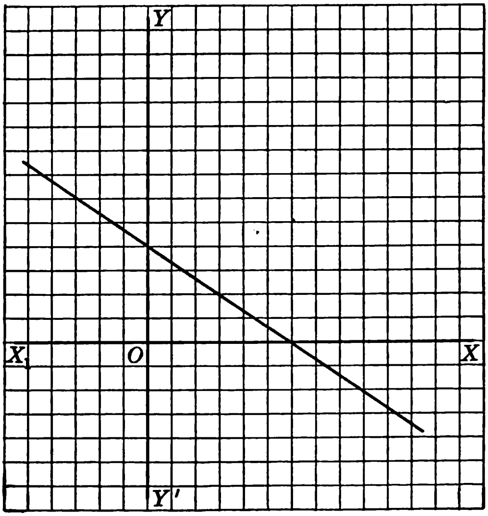

FIG. 19.

The graph of this equation is given in Figure 19. This result illustrates the general truth that **the graph of an equation of the first degree in two variables is a straight line.** For this reason such an equation is called a **linear equation.**

The graph of an equation is usually called the **locus** of the equation.

Since only two points are necessary to locate a straight line, how may the locus of a linear equation be easily drawn?

EXAMPLE. Find the locus of the equation $x - 2y = 8$.

SOLUTION. We can find two points on this locus most easily by first letting $y = 0$, which gives $x = 8$; and then letting $x = 0$, which gives $y = -4$. Let the student draw the locus.

Exercise 58

Construct the locus of the following equations:

1. $x + y = 4$.
2. $3x - y = 5$.
3. $y - 4x = 6$.
4. $2x = 3y$.
5. $x + 12y = 4$.
6. $x = 6$.
7. $7x + 4y = 10$.
8. $x + y = 0$.
9. $y = -5$.

53. Graphical solution of equations. If we construct the loci of two equations, as $x + y = 5$ and $2x - y = 4$, as shown in Figure 20, it is seen that the point of intersection is (3,2). As this point is on the locus of both equations the values $x = 3$, $y = 2$, will satisfy both equations. Check by substitution. A pair of values such as (3,2) which satisfies each of two equations is called a **solution** of the pair of equations. Since the graph of a linear equation is a straight line, and since two straight lines may intersect in only one point, there is in general only one pair of values of x and y that satisfies a pair of linear equations in x and y.

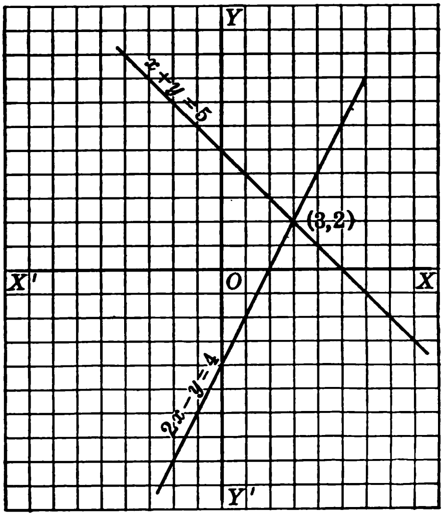

FIG. 20.

Exercise 59

Plot the loci of the following equations, and find the solution of each pair from the graph. Test each solution by substitution in the equations.

1. $x + y = 4,$
$x - y = 6.$

2. $x - y = 2,$
$2x + 3y = 14.$

3. $3x - y = 0,$
$2x + 5y = 0.$

4. $6x - y = 5,$
$x + 2y = 3.$

5. $x + 2y = 4,$
$x + y = 7.$

6. $6x - y = 4,$
$6x - 2y = 5.$

7. $x = 5,$
$y = 3.$

8. $5x - 2y = 8,$
$y = 1.$

9. $2x + 2y = 5,$
$2x + 4y = 10.$

10. $x = 3,$
$3y = 8.$

11. Plot the locus $y - 3x = 4$, then multiply both members of the equation by 3, and plot the locus of the resulting equation. Compare the two graphs.

12. Plot the loci of $y + 2x = 5$, and of $3x - 2y = 4$. Then form an equation by adding the members of these two equations and plot the locus of the resulting equation. What are the coördinates of the point common to the three loci? Plot the loci of $x = 2$ and $y = 1$.

54. Graphs of quadratic functions. In making the graph of a linear equation, y could be expressed as a function of x. This function of x involved only the first power of x.

A function of a variable which involves the second power of the variable and no higher power is called a **quadratic function.** If $y = 2x^2 + 3x + 5$, then y is a quadratic function of x.

The graph of a quadratic function can be made by giving values to x, finding the corresponding values for y, and then plotting the points having these values of x and y as coördinates.

EXAMPLE. Make the graph of $3x^2 - 2$.

SOLUTION

$y = 3x^2 - 2$.

x	y
-2	10
$-1\frac{1}{3}$	$\frac{10}{3}$
-1	1
$-\frac{2}{3}$	$-\frac{2}{3}$
$-\frac{1}{3}$	$-\frac{5}{3}$
0	-2
$\frac{1}{3}$	$-\frac{5}{3}$
$\frac{2}{3}$	$-\frac{2}{3}$
1	1
$\frac{4}{3}$	$\frac{10}{3}$
2	10

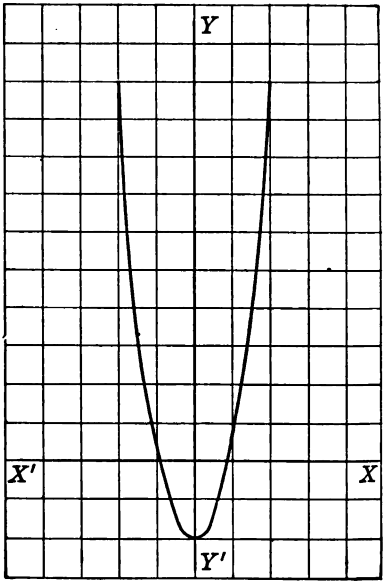

FIG. 21.

Exercise 60

Make graphs of the following functions:

1. x^2.
2. $x^2 + 1$.
3. $x^2 + x$.
4. $2x^2 - 12$.
5. $x^2 - 5x$.
6. $3x^2 - x - 5$.
7. $-x^2$.
8. $8 + 5x - x^2$.
9. $4x - x^2$.

Exercise 61

1. Plot the lines, $x = 4$, $y = 2$. Give the coördinates of their point of intersection. Give the equations of two lines that intersect in the point (4,7); two that intersect in the point(−2,6,); two that intersect in the point (8,−5).

2. If 1000 cm.^{3} of mercury with the temperature of 0° centigrade are taken, then the volume, V, in cubic centimeters of the mercury at any temperature t between 0° and

40° centigrade is given by the formula $V = 1000 + .18t$. Plot this equation from $t = 0°$ to $t = 40°$. From the graph find the increase in the volume from 0° to 20°; from 20° to 30°; from 30° to 40°.

3. On railroad curves the outside rail is elevated above the inside rail. The elevation, e, in inches, is given by the formula $e = \frac{gv^2}{32R}$, where g = the gage or width of the track in feet, v = the velocity of the train in feet per second, and R = the radius of the curve in feet. The standard gage for American railroads is 4 ft. 8½ in. What should be the elevation of the outer rail on a standard gage American road to allow for a speed of 40 miles an hour on a curve of radius 1000 ft.? For a given speed and gage, will the elevation increase or decrease as the radius of the curve increases? With a given gage and radius, what must be done with the elevation to allow for doubling the speed?

4. Plot the graphs of the equations

$$\begin{aligned} &\text{(a)}\quad x - 2y - 4 = 0, \\ &\text{(b)}\ 2x + \ \ y - 5 = 0, \\ &\text{(c)}\ 2x - \ \ y - 6 = 0. \end{aligned}$$

Do these lines appear to meet at a point? Have the three equations a common solution? Find the coördinates of the point or points in which the lines meet.

5. The formula $h = 800t - 16t^2$ gives the height in feet at the end of t seconds of a bullet shot vertically upward with a velocity of 800 ft. per second. Find values of h for values of t that are multiples of 5 sec. from 0 sec. up to 50 sec., and plot the graph of the equation using t as abscissa and h as ordinate. Find the height at the end of 10 sec.; of 20 sec. After how many seconds has the bullet reached its highest point? How many seconds before the bullet will reach the ground? Will it take as long for it to fall as to rise? This formula neglects the resistance of the air.

CHAPTER VIII

SYSTEMS OF LINEAR EQUATIONS

55. Solution of an equation in two unknowns. The equation $2n + 5 = 3n - 4$ is satisfied by only one value of the unknown. What is that value? The equation $x^2 - x = 6$ is satisfied by the values $x = -2$ and $x = 3$. Prove it. If an equation has two unknowns, a pair of values, one for each unknown, is required to satisfy the equation. This pair of values is called one **solution** of the equation. In the equation $x + y = 7$, one solution is $x = 2$, $y = 5$. Find another solution. How many solutions can you find? Find the value of y corresponding to $x = -3$; to $x = 2\frac{1}{2}$.

These facts should be clear: (1) Not every pair of numbers will satisfy a particular equation in two unknowns; (2) A great many solutions can be found for an equation in two unknowns; (3) Any number may be chosen as the value of one of the unknowns and a corresponding value found for the other unknown.

Write ten solutions of the equation $x + y = 9$. Do any of the solutions of this equation which you have written also satisfy the equation $x - y = 5$? Write ten solutions for each of the following equations: $2x + y = 13$, $x + 2y = 11$. Have you written any pair of values which satisfies both equations? Such a pair of equations containing the same unknowns is called a **system of equations.** We shall consider the solutions of such systems.

56. Elimination by addition and subtraction. Let us suppose that the equations (1) $3x - 2y = 6$, and (2) $2x + y = 11$ have a common solution, and discover a way to find it.

In this case x represents the same number in the two equations and y represents the same number.

Multiply equation (1) by 2. $(3)\ 6x - 4y = 12.$
Multiply equation (2) by 3. $(4)\ 6x + 3y = 33.$
Subtract (3) from (4). $7y = 21.$
Divide (5) by 7. $y = 3.$
Substitute 3 for y in (1). $3x - 6 = 6.$
Transpose and collect. $3x = 12.$
Divide by 3. $x = 4.$

Then $x = 4$, and $y = 3$ is one solution of the given equations. Let the pupil check the solution.

In this method of solving the equations the aim was to get an equation containing only one of the unknowns. This is done by eliminating one of the unknowns. To eliminate one of the unknowns, x in this case, the terms containing that unknown were made to have the same coefficient in the two equations How was it decided by what to multiply each equation so that those terms would have the same coefficient? When shall the equations be subtracted and when added to eliminate the one unknown?

EXAMPLE 1. Solve the system of equations,

$$(1)\ 5x + 6y = 21,$$
$$(2)\ 8x - 9y = 15.$$

Decide which unknown to eliminate. In this case choose y. Why? Make the terms containing y have the same coefficient in the two equations. This coefficient is the least common multiple of the coefficients of y in the two equations. By what must (1) be multiplied to get an equation having 18 as the coefficient of y? By what must (2) be multiplied to get an equation having the same coefficient for y?

Multiply (1) by 3. $(3)\ 15x + 18y = 63.$
Multiply (2) by 2. $(4)\ 16x - 18y = 30.$
Add (3) and (4). Why? $(5)\ 31x = 93.$
Divide (5) by 31. $x = 3.$

How can the corresponding value of y be found? Test these values in both (1) and (2).

Exercise 62

Solve and check the following:

1. $m + n = 5,$
$m - n = 1.$

2. $2h + a = 10,$
$h + 2a = 6.$

3. $2x + y = 21,$
$3x - y = 4.$

4. $4r + s = 25,$
$5r - 2s = 35.$

5. $7s = 2t - 15,$
$s + 6 = t.$

6. $5p + 9q = 8,$
$6q = 9p - 7.$

7. $5a - 5b = 15,$
$3b = 71 - 5a.$

8. $2c + d - 9 = 0,$
$5c = 25 - 3d.$

9. $\frac{n}{2} - \frac{m}{3} = 6,$
$\frac{3n}{2} + \frac{2m}{3} = 8.$

10. $\frac{2a}{3} + y + 7 = 0,$
$5a - 3y = 105.$

11. $\frac{(3x - 8y)}{2} - 6x = 9,$
$y - \frac{1}{4} = \frac{3x}{4}.$

12. $5h + .7k = -1.25,$
$.12h - .08k = 3.6.$

13. $\frac{.1x}{2} + \frac{.1y}{3} = \frac{1.3}{3}, \; .1x = .76 + \frac{.2y}{5}.$

14. Solve for x and y in terms of a and b:
$x + y = a, \; x - y = b.$

15. Solve for r and s:
$a = \frac{(r + s)}{2}, \; r - s = b.$

16. Solve for m and n:
$m + n = a, \; 2n = b.$

17. Solve for h and k:
$h + k = a + b, \; h - k = a - b.$

18. Solve for a and b:
$6a - b = 5, \; a = 2.$

57. Elimination by substitution. When two linear equations in two unknowns are given, it is always possible to eliminate one of the unknowns by solving one of the equations for that unknown in terms of the other, and substituting that value for the unknown in the other equation. This is a very useful plan for it can sometimes be used when the equations are of higher degree than the first.

EXAMPLE 1. Solve the equations, (1) $x - 2y = 3$,
(2) $2x - y = 12$.

SOLUTION.

Solve (1) for x. (3) $x = 2y + 3$.
Substitute $2y + 3$ for x in (2). $2(2y + 3) - y = 12$.
Solve for y. $4y + 6 - y = 12$.
$3y = 6$.
$y = 2$.
Substitute 2 for y in (3), $x = 4 + 3 = 7$.

Solve the same system of equations by solving (2) for y in terms of x and substituting this value for y in equation (1). Check your solution by substituting the values in both the original equations.

EXAMPLE 2. Eliminate a from the system of equations, (1) $\frac{1}{2}at^2 = s$, (2) $2as = v^2$.

SOLUTION.

Solve (2) for a. $a = \frac{v^2}{2s}$.

Substitute in (1). $\frac{1}{2}\cdot\frac{v^2}{2s}\cdot t^2 = s$.

Simplify. $\frac{v^2t^2}{4s} = s$.

Clearing of fractions, $v^2t^2 = 4s^2$, an equation from which a is eliminated. These formulas state important laws of motion, explained in physics.

EXAMPLE 3. Solve the equations, (1) $\frac{x}{3} - \frac{y}{4} = 2$, and (2) $\frac{2x}{5} + \frac{3y}{6} = 1\frac{3}{5}$.

SOLUTION.

Multiply (1) by 12. $4x - 3y = 24$.
Multiply (2) by 30. $12x + 15y = 48$.

Eliminate either by addition or substraction or by substitution, and complete the solution.

Exercise 63

Solve the following systems, eliminating by substitution:

1. $x - y = 3$,
$x + y = 7$.

2. $y - 2x = 6$,
$x + 2y = -8$.

3. $2x + 5y = 8$,
$x - 10y = 9$.

4. $3m = 13 + 8n$,
$m + 6n = 0$.

5. $\frac{3a}{4} - \frac{b}{12} = 3\frac{1}{2}$,
$a + 2b + 8 = 0$.

6. $h + \frac{k}{7} = 6$,
$\frac{h}{4} - \frac{k}{12} = \frac{2}{3}$.

7. $.2x - .1y = \frac{1}{2}$,
$5x - 14 = 2y$.

8. $2t = 7$,
$t^2 = 12s^2 + 1$.

9. $a - b = 1$,
$a^2 = 2b + 5$.

In the following eliminate either by addition or subtraction or by substitution as is more convenient.

10. $\frac{a}{2} - \frac{b}{3} = 5$,
$\frac{a}{7} + \frac{b}{6} = 3$.

11. $\frac{5m}{6} + \frac{d}{4} = 7$,
$\frac{2m}{3} - \frac{d}{8} = 3$.

12. $x = 2y$,
$\frac{3x}{4} + 2\frac{1}{2}y = 8$.

13. $\frac{3x + y}{4} = 5$,
$\frac{5x}{6} = y + 4\frac{1}{3}$.

14. $C = 2\pi r$ and $d = 2r$. Eliminate r from these two equations, thus expressing c in terms of d.

15. $A = \pi r^2$ and $d = 2r$. Eliminate r from these two equations, expressing A in terms of d.

16. In an equilateral triangle the area, A, equals one-half the side, a, times the altitude h; and the side equals one-half the altitude times the square root of 3. Express these statements as formulas, then eliminate h, thus expressing the area in terms of the side a.

58. Problems involving two unknowns. The pupil has already learned to solve problems involving two, and sometimes more than two unknowns, but he has done it by expressing all the unknowns in terms of one of them. It is often much easier to use two letters to represent two of the unknowns and to express the other unknowns in terms of these two. It is then necessary to get two equations from the conditions of the problem so that one of the two unknowns can be eliminated and the other found.

Example. The circumference of the hind wheel of a wagon is 2 feet more than the circumference of the fore wheel. The hind wheel lacks two feet of going as far in 4 revolutions as the front wheel goes in 5. Find the circumference of each wheel.

Solution. Let C = the number of feet in circumference of the hind wheel,
and c = the number of feet in circumference of the fore wheel.

Then (1) $C - c = 2$,
and (2) $5c = 4C + 2$.
Transposing in (2), (3) $4C - 5c = -2$.
Multiplying (1) by 5, (4) $5C - 5c = 10$.
Subtracting (3) from (4), $C = 12$.
Substituting 12 for C in (1), $12 - c = 2$.
Solving for c, $c = 10$.
Check. $12 - 10 = 2$.
$50 - 48 = 2$.

Exercise 64

1. The sum of two numbers is 7. Their difference is 1. What are the numbers?

2. The average of two numbers is 12. The larger is 3 more than twice the smaller. Find the numbers.

3. The length of a certain rectangle is 3 ft. more than $\frac{3}{5}$ of the width. The perimeter is 22 ft. Find its length and width.

4. How many cents are 3 quarters and 7 dimes worth? How many cents are x dimes and y cents worth? If you

have $2 in dimes and quarters and have 11 coins in all, how many dimes and how many quarters have you?

5. At a certain entertainment the price of admission for children was 15¢, for adults 25¢. When all had been admitted the doorkeeper found that he had admitted 234 persons and had taken in $53.10. How many of the audience were children?

6. A newsboy sells the "Post" and the "Times." One day he sold 70 papers and found that he had taken in $1.72. He had charged 3¢ for the "Post" and 2¢ for the "Times." How many of each paper had he sold?

7. A dairyman wishes to mix 3% milk and 5% milk to get 10 gallons of 4.5% milk. How much of each quality of milk shall he use?

8. Suppose the dairyman wishes to mix a% milk and b% milk to get n pounds of c% milk. How many pounds of each quality must he use? The answer to this problem is a formula by which all such problems may be solved. Use it to solve problem 7.

9. How much 3.5% milk must be mixed with 5% milk to get 100 pounds of 4% milk? Use the formula of problem 8.

10. If two trains start from the same station and run in opposite directions for 3 hours they are 210 miles apart. If they run in the same direction for 4 hours they are 40 miles apart. What is the speed of each?

11. Two angles are complementary. The smaller is 6° more than one-fifth of the larger. Find the two angles.

12. Two angles are supplementary. One is 16° larger than the other. Find the two angles.

13. A number contains two digits. The tens' digit is 3 more than the units' digit. The number is 7 times the sum of the digits. Find the number.

14. A number contains two digits. The sum of the digits is 12. If the digits are reversed the resulting number is 18 more than the given number. Find the given number.

59. Inconsistent equations. Find a solution for the equations $x + y = 5$ and $x + y = 7$. Can you eliminate x or y by subtraction? Do you think that there is a pair of numbers that will satisfy both equations?

Let us see if we can get any information about a solution of these equations from their graphs. The graphs are given in Figure 22. The graphs are parallel lines. Since the lines have no common point the equations have no common solution.

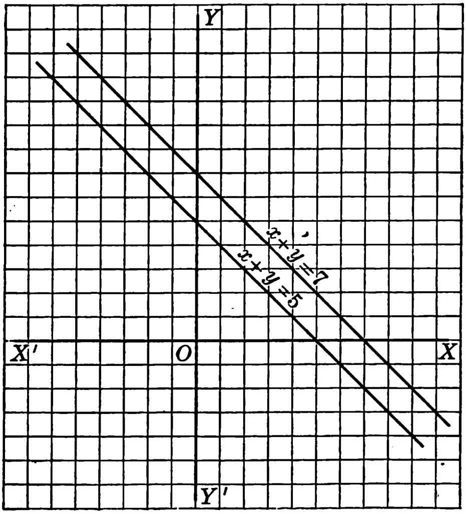

FIG. 22.

Two equations are said to be **inconsistent** if they have no common solution.

60. Equivalent equations. Can you find a common solution for the equations $x + 2y = 3$ and $3x + 6y = 9$? Try solving them. Construct their graphs. What is your conclusion?

When you construct the graphs you will find that they coincide. Therefore any solution of one equation is also a solution of the other. Find three solutions of $x + 2y = 3$

and see if they are solutions of $3x + 6y = 9$. Can you perform any operation upon the equation $x + 2y = 3$ so that the result will be the equation $3x + 6y = 9$? Can you perform any operation upon the equation $3x + 6y = 9$ so that the result will be the equation $x + 2y = 3$?

Equations are said to be **equivalent** or **dependent** if they have the same graph. Any solution of one is also a solution of the other.

Two equations that are neither inconsistent nor equivalent are said to be **independent.**

We have now seen that two linear equations in two variables may

(a) Have just one common solution. In that case the graphs intersect, and the equations are independent.

(b) Have no common solution. In that case the graphs are parallel lines, and the equations are inconsistent.

(c) Have all of their solutions in common. The graphs then coincide, and the equations are equivalent.

Exercise 65

Construct the graphs of each of the following systems of equations. Tell what common solutions these equations have, and whether the equations are independent, inconsistent, or equivalent.

1. $x - y = 4,$
$2x - 2y = 7.$

2. $2x + 3y = 12,$
$6x + 9y = 36.$

3. $2x + 3y = 12,$
$x - y = 8.$

4. $2x + 3y = 12,$
$4x + 6y = 10.$

5. $x = 5,$
$3x = 7.$

6. $x = 1,$
$y = 2.$

7. $x - y = 0,$
$x + y = 0.$

8. $3x - y = 4,$
$y - 3x + 4 = 0.$

9. $x = y,$
$-x = -y.$

10. $x = 2y,$
$x - 2y = 6.$

Exercise 66

1. Write a number in which a is the units' digit, b the tens' digit, c the hundreds', and d the thousands'. Write another number having the same digits. Show that the difference of these two numbers is divisible by 9. Will this be true of every two numbers having the same digits? Make the same inquiry for a number of six digits.

2. Solve the formula $l = a + (n - 1)d$ for n; for d.

3. Solve the formula $s = \frac{n}{2}(a + l)$ for n; for a, for l.

4. The sum of the bases of a certain trapezoid is 11 inches more than its altitude. The area is 21 sq. in. Find the altitude.

5. Find the square root of $n^2 - n + \frac{1}{4}$.

6. Represent by a circular pictogram the number of boys and the number of girls in your class.

7. What three ways have been given of expressing the relations that exist between quantities?

8. Plot the points whose coördinates are these pairs of values of x and y:

x	−2	0	1	3	5	6
y	−9	−5	−3	1	5	7

9. From the graph of the function $F = \frac{9}{5}C + 32$ find the temperature at which the centigrade and Fahrenheit readings are the same.

10. A cylindrical steel bar l feet long and r inches in radius is melted and cast into balls r' inches in radius. Make a formula that gives the number, N, of balls that the cylinder will make.

11. Find the number of steel balls in problem 10 if $r =$ 3 in., $l = 6$ ft., and $r' = \frac{1}{4}$ in.

12. A sphere 2 ft. in diameter is cut from a cylinder 2 ft. in diameter and 2 ft. high. The volume of the sphere is what fraction of the volume of the cylinder? The surface of the sphere is what fraction of the surface of the cylinder?

CHAPTER IX

EXPONENTS AND RADICALS

61. Square root. A **square root** of a number is one of its two equal factors. Thus, 4 is a square root of 16, since $4 \cdot 4 = 16$.

Since $(+x)\ (+x) = (-x)\ (-x) = x^2$, it follows that every square has two square roots differing only in sign. Thus, the square roots of 36 are $+6$ and -6.

A **cube root** of a number is one of its three equal factors. For example, a cube root of 27 is 3, and a cube root of x^6 is x^2. In general we say that an *n*th **root** of a number is one of its n equal factors.

The **radical sign** $\sqrt{\ }$, is used to indicate the positive square root of the number under it. When a negative square root is intended the radical sign is preceded by the sign $-$.

Thus, $\sqrt{25}$ means $+5$, and $-\sqrt{25}$ means -5.

An indicated root is called a **radical.**

For example, $\sqrt{x}$ is a radical.

By means of a figure called the **index of the root** the radical sign is used to indicate other roots than square roots.

Thus, $\sqrt[3]{a}$ indicates the cube root of a, and $\sqrt[n]{a}$ indicates the nth root of a.

The square root of a number can be found at once if we can separate the number into two equal groups of factors. Thus, $\sqrt{225} = \sqrt{3 \cdot 3 \cdot 5 \cdot 5} = \sqrt{(3 \cdot 5)\ (3 \cdot 5)} = \sqrt{15 \cdot 15} = 15.$

$$\begin{aligned} \text{Also, } \sqrt{121a^4b^2} &= \sqrt{11 \cdot 11 \cdot a \cdot a \cdot a \cdot a \cdot b \cdot b} \\ &= \sqrt{(11 \cdot a \cdot a \cdot b) \cdot (11 \cdot a \cdot a \cdot b)} \\ &= \sqrt{11a^2b \cdot 11a^2b} = 11a^2b. \end{aligned}$$

Exercise 67

Find the following indicated roots:

1. $\sqrt{9}$.

2. $\sqrt{49}$.

3. $\sqrt{64}$.

4. $-\sqrt{81}$.

5. $-\sqrt{144}$.

6. $\sqrt{256}$.

7. $\sqrt{196}$.

8. $\sqrt{289}$.

9. $\sqrt{1225}$.

10. $-\sqrt{400}$.

11. $-\sqrt{10000}$.

12. $\sqrt{27^2}$.

13. $\sqrt{x^2y^2}$.

14. $-\sqrt{49b^2}$.

15. $-\sqrt{289m^4}$.

16. $\sqrt[3]{27}$.

17. $-\sqrt[3]{125}$.

18. $\sqrt[3]{x^6}$.

19. $\sqrt[3]{64b^3c^6}$.

20. $-\sqrt[3]{8^6}$.

62. Square root of a product. We know that $\sqrt{4\cdot 25} = \sqrt{4}\cdot\sqrt{25}$ since each equals 10. It is true in general that

$$\sqrt{ab} = \sqrt{a}\sqrt{b}.$$

This formula gives the following

Rule. *The square root of a product equals the product of the square roots of the factors.*

This principle enables us to reduce radicals to more convenient forms for computation in many cases. One important case is that of expressing the square roots of numbers in terms of the square roots of smaller numbers, and expressing the square roots of fractions in terms of the square roots of whole numbers. This reduction is called **simplification of radicals.**

EXAMPLE 1. Simplify $\sqrt{50}$.

SOLUTION. $\sqrt{50} = \sqrt{25\cdot 2} = \sqrt{25}\cdot\sqrt{2} = 5\sqrt{2}$.

If then the pupil has memorized $\sqrt{2}$, he can say immediately that

$$\sqrt{50} = 5 \times 1.4142 = 7.071.$$

EXAMPLE 2. Simplify $\sqrt{12a^3b}$.

SOLUTION. $\sqrt{12a^3b} = \sqrt{4a^2\cdot 3ab} = 2a\sqrt{3ab}$.

The steps in each of the preceding solutions are:

1. Separating the number into two factors one of which is the product of all the factors which are squares.

2. Finding the square root of this square and multiplying the remaining radical by it.

Exercise 68

Simplify the following, leaving no square factor under the radical sign.

1. $\sqrt{12}$.
2. $\sqrt{72}$.
3. $\sqrt{28}$.
4. $\sqrt{200}$.
5. $\sqrt{96}$.
6. $\sqrt{500}$.
7. $\sqrt{98}$.
8. $\sqrt{192}$.
9. $\sqrt{20}$.
10. $\sqrt{75}$.
11. $\sqrt{x^3}$.
12. $\sqrt{a^2b}$.
13. $\sqrt{3m^2n^2}$.
14. $\sqrt{24ab^3}$.
15. $\sqrt{80r^2s}$.
16. $-\sqrt{6^2 \cdot 7^3}$.
17. $-\sqrt{300rs}$.
18. $-\sqrt{3^3a^3}$.
19. $\sqrt{7ab}$.
20. $\sqrt{a^4}$.
21. $\sqrt{\pi^2r^2}$.
22. $3\sqrt{25}$.
23. $a\sqrt{100b^2c^2}$.
24. $6\sqrt{45}$.
25. $2\sqrt{36a}$.
26. $4\sqrt{50a^2}$.
27. $a\sqrt{ab^2}$.
28. $\sqrt{100a^6b^4}$.
29. $\sqrt{343a^6b^3}$.
30. $\sqrt{125m^3n^{12}}$.

63. Importance of memorizing certain square roots. The $\sqrt{2}$, 1.4142, and the $\sqrt{3}$, 1.7321, occur frequently in practical problems and should be memorized. Using the values of $\sqrt{2}$ and $\sqrt{3}$ find the values of the following expressions correct to three decimal places, without further extraction of roots.

1. $\sqrt{12}$.
2. $\sqrt{18}$.
3. $\sqrt{72}$.
4. $\sqrt{50}$.
5. $\sqrt{98}$.
6. $\sqrt{2} + \sqrt{3}$.
7. $\sqrt{8} + \sqrt{9}$.
8. $\sqrt{32} + \sqrt{108} - \sqrt{144}$.
9. $7\sqrt{2} - 5\sqrt{3} + 15$.
10. $6\sqrt{3} - \sqrt{3}$.
11. $\frac{1}{2}\sqrt{3}$.
12. $\frac{3}{4}\sqrt{2}$.
13. $2 - \sqrt{2}$.
14. $\frac{\sqrt{2}}{2}$.
15. $\sqrt{72} - \sqrt{36}$.
16. $3\sqrt{3} - 3$.

64. Square roots of fractions. A fraction is squared by squaring the numerator and squaring the denominator.

Thus, $(\frac{4}{7})^2, = \frac{4}{7} \cdot \frac{4}{7} = \frac{16}{49}$.

Hence to find the square root of a fraction we may take the square root of the numerator and the square root of the denominator.

For example, $\sqrt{\frac{16}{49}} = \frac{4}{7}$.

By this method, finding $\sqrt{\frac{2}{3}} = \frac{1.4142}{1.7321} = .816$, correct to 3 decimal places requires finding two square roots and a division of one five-place number by another five-place number. Much time and labor may be saved by either of the following methods:

(a) Reducing the common fraction to a decimal.

By this method, $\sqrt{\frac{2}{3}} = \sqrt{.66666666} = .816$.

(b) Multiplying both numerator and denominator by a number that makes the denominator a square.

This is called rationalizing the denominator.

$$\text{Thus,}\qquad \sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{\sqrt{9}} = \frac{2.449}{3} = .816.$$

In the following exercises use the method of finding the square root that is simplest and most convenient.

Exercise 69

Find the following square roots correct to two decimal places.

1. $\sqrt{\frac{4}{9}}$.
2. $\sqrt{\frac{25}{36}}$.
3. $\sqrt{\frac{144}{400}}$.
4. $\sqrt{\frac{35}{64}}$.
5. $\sqrt{\frac{8}{32}}$.
6. $\sqrt{\frac{19}{100}}$.
7. $\sqrt{\frac{1}{2}}$.
8. $\sqrt{\frac{1}{3}}$.
8. $\sqrt{.1}$.
10. $\sqrt{.25}$.
11. $\sqrt{.64}$.
12. $\sqrt{.45}$.
13. $\sqrt{\frac{5}{3}}$.
14. $\sqrt{\frac{9}{2}}$.
15. $\sqrt{\frac{15}{25}}$.
16. $\sqrt{.0025}$.
17. $\sqrt{2.5}$.
18. $\sqrt{.001}$.
19. $\sqrt{\frac{5}{6}}$.
20. $\sqrt{\frac{5}{18}}$.

65. Multiplication of radicals. By using the formula

$$\sqrt{a}\sqrt{b} = \sqrt{ab},$$

we have a method of finding the product of two radicals.

EXAMPLE 1. Multiply $\sqrt{6}$ by $\sqrt{8}$ and simplify the result.

SOLUTION. $\sqrt{6}\cdot\sqrt{8} = \sqrt{48} = \sqrt{16\cdot 3} = 4\sqrt{3}.$

EXAMPLE 2. Multiply $\sqrt{5ab}$ by $\sqrt{10a^2b}$, and simplify the result.

SOLUTION. $\sqrt{5ab}\cdot\sqrt{10a^2b} = \sqrt{50a^3b^2} = \sqrt{25a^2b^2\cdot 2a} = 5ab\sqrt{2a}.$

Exercise 70

Find the products of the following radicals and simplify the results.

1. $\sqrt{5}\cdot\sqrt{5}$.
2. $\sqrt{2}\cdot\sqrt{8}$.
3. $\sqrt{6}\cdot\sqrt{24}$.
4. $\sqrt{a}\cdot\sqrt{a}$.
5. $\sqrt{a}\cdot\sqrt{ab^2}$.
6. $\sqrt{5a}\cdot\sqrt{5a}$.
7. $\sqrt{m}\cdot\sqrt{4m}$.
8. $\sqrt{\pi}\cdot\sqrt{\pi r^2}$.
9. $\sqrt{2}\cdot\sqrt{6}$.
10. $\sqrt{3}\cdot\sqrt{6}$.
11. $\sqrt{6}\cdot\sqrt{12}$.
12. $\sqrt{8}\cdot\sqrt{18}$.
13. $\sqrt{8}\cdot\sqrt{30}$.
14. $\sqrt{3}\cdot\sqrt{27}$.
15. $\sqrt{x^2y}\cdot\sqrt{xy}$.
16. $\sqrt{4ab}\cdot\sqrt{9a^2b^2}$.
17. $\sqrt{\frac{3}{5}}\cdot\sqrt{\frac{4}{15}}$.
18. $\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{3}{4}}$.
19. $\sqrt{\frac{a}{b}}\cdot\sqrt{\frac{b}{c}}\cdot\sqrt{c^2}$.
20. $\sqrt{12}\cdot\sqrt{10}\cdot\sqrt{3}$.
21. $\sqrt{5}\cdot\sqrt{\frac{2}{5}}\cdot\sqrt{\frac{3}{4}}$.
22. $\sqrt{2ab}\cdot\sqrt{6a^2b^3}\cdot\sqrt{3a}$.
23. $\sqrt{a+b}\cdot\sqrt{a+b}$.
24. $\sqrt{r-s}\cdot\sqrt{r-s}$.
25. $\sqrt{a}\cdot\sqrt{\frac{4}{a}}$.
26. $\sqrt{\frac{ac}{b}}\cdot\sqrt{\frac{ab}{c^2}}\cdot\sqrt{c}$.
27. $\sqrt{.1}\cdot\sqrt{.001}$.
28. $\sqrt{6xy^2}\cdot\sqrt{5x}$.
29. $\sqrt{12abc}\cdot\sqrt{9a^2b}$.

66. Laws of exponents. The student has already used the following laws:

I. $a^m \cdot a^n = a^{m+n}$. Law for multiplication.
II. $a^m \div a^n = a^{m-n}$. Law for division.

The student should state each of these laws in words.

The expression $(a^2)^3$ means that a^2 is to be used as a factor 3 times. That is, $(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^{2+2+2} = a^6$.
In the same way, $(a^m)^n$ means that a^m is to be taken as a factor n times. That is, $(a^m)^n = a^m \cdot a^m \cdot a^m \cdots\cdot$ to n factors $= a^{m+m+m+\cdots \text{ to } n \text{ terms}} = a^{mn}$.

This is the law of exponents for raising to powers.

In finding $\sqrt{a^6}$ we may write

$$\sqrt{a^6} = \sqrt{a^3 \cdot a^3} = a^3 = a^{\frac{6}{2}}.$$

In general if n is an integer, then $2n$ is an even integer, and we may write

$$\sqrt{a^{2n}} = a^n = a^{\frac{2n}{2}}.$$

That is, to take the square root of an even power of a number, we divide the exponent of the power by 2.

Similarly,

$$\sqrt[3]{a^{12}} = \sqrt[3]{a^4 \cdot a^4 \cdot a^4} = a^4 = a^{\frac{12}{3}}.$$

Make a rule for finding the cube root of a power of a number; for finding the fourth root of a power of a number; for finding the nth root of a power of a number if the exponent of the power is divisible by n.

We are thus led to the formula,

$$\sqrt[n]{a^m} = a^{m \div n},$$

if m is divisible by n. This is the law of exponents in finding roots.

The student should state this law in words.

State the law $(a^m)^n = a^{mn}$ in words.

Exercise 71

Perform the following indicated operations. Reduce fractions to lowest terms.

1. $a^4 \cdot a^6$.
2. $x^3 \cdot x^6 \cdot x$.
3. $2m^2n^5 \cdot 2mn^2$.
4. $6rs^3 \cdot 8r^2s$.
5. $(a^3)^2$.
6. $(xy^3)^3$.
7. $(2ab^4)^2$.
8. $(b^n)^3$.
9. $(x^ny^2)^2$.
10. $(-3m^2n^3)^3$.
11. $(-2abc)^3$.
12. $x^n \cdot x^3 \cdot x$.
13. $h^n \cdot 2h^{2n} \cdot h$.
14. $5^2 \cdot 5^3 \cdot 5$.
15. $2 \cdot 3^2 \cdot 2 \cdot 3$.
16. $(-5)^2 \cdot (-5^2)$.
17. $a^5 \div a^3$.
18. $m^6 \div m^2$.
19. $r^{14} \div r^7$.
20. $m^4n^6 \div mn^2$.
21. $k^n \div k^2$.
22. $s^{3n} \div s^3$.
23. $b^{n+3} \div b^3$.
24. $a^nb^4 \div ab$.
25. $\frac{6a^3b^2}{2ab}$.
26. $\frac{12m^6n^3}{4mn^3}$.
27. $\frac{a^3b^2}{a^5b}$.
28. $\frac{z^3}{5yz^3}$.
29. $\frac{7rs^2}{5r^2s^2}$.
30. $\frac{a^{3n}b}{a^3b^3}$.
31. $\frac{a^{n+2}}{a^{n-2}}$.
32. $\sqrt{a^4b^2}$.
33. $\sqrt{16m^2n^4}$.
34. $\sqrt{25x^2y^2z^4}$.
35. $\sqrt{a^{2n}}$.
36. $\sqrt[3]{x^3y^6}$.
37. $\sqrt[3]{27m^6n^3}$.
38. $\sqrt[3]{125s^{12}}$.
39. $\sqrt{\frac{36v^2}{49m^8}}$.
40. $\sqrt[3]{\frac{64r^3}{1000m^6}}$.
41. $\sqrt[3]{\frac{343x^{3n}}{125y^6}}$.
42. $\sqrt[n]{\frac{5^nx^{2n}}{3^{2n}y^n}}$.
43. $\sqrt{a^2 + 2ab + b^2}$.
44. $\sqrt{x^2 - 2xy + y^2}$.
45. $\sqrt{x^2 - 69 + 9}$.
46. $\sqrt{(x + y + z)^2}$.
47. $\sqrt{(x + y)^4}$.
48. $\sqrt{(x - 4)^2(x + 6)^2}$.

Exercise 72

The answers to the first 12 problems are to be left in the radical form. Simplify, leaving no square factor under the radical sign.

1. The side of a square is a. Find the diagonal.

2. Find the diagonal of a square whose side is 35.

3. Find the side of a square whose area is 2000.

4. Write a formula for finding the side, s, of a square whose area is A.

5. Write a formula for finding the side, s, of a square whose area is equal to that of a circle of radius r.

6. What is the diagonal of a rectangle whose adjacent sides are a and $2a$?

7. Find the diagonal of a room whose height is h, width $2h$, and length $3h$.

8. Make a formula for finding the radius of a sphere whose surface is S.

9. Make a formula for finding the radius of a sphere whose volume is V.

10. Draw an equilateral triangle ABC. Let s equal the length of one side. Draw a perpendicular BD from the vertex B to the opposite side. Let $BD = h$. What is the length of AD? Make a formula for finding h when s is given. Make a formula for finding the area of the triangle when the side is known.

11. Find the altitude of an equilateral triangle whose side is 10. Find its area.

12. Solve the formula $s = \frac{1}{2}gt^2$ for t.

In the next four problems r is the radius of a circle, and s is the side of an inscribed polygon. Find the numerical answers correct to .001.

13. In a regular inscribed triangle, $s = r\sqrt{3}$. Find s when $r = 10$; when $r = 25$.

14. In a regular inscribed square, $s = r\sqrt{2}$. Find s when $r = 10$; when $r = 25$.

15. In a regular inscribed hexagon, $s = r$. Find s when $r = 10$; when $r = 25$.

16. In a regular inscribed decagon, $s = \frac{(\sqrt{5} - 1)r}{2}$. Find s when $r = 10$; when $r = 25$.

17. In a circle of given radius, does the length of the side of the regular inscribed polygon increase or decrease as the number of sides of the polygon increases? Consider the answers in the four preceding exercises.

18. Find the perimeter of the triangle in problem 13 when $r = 10$.

19. Find the perimeter of the square in problem 14 when $r = 10$.

20. Find the perimeter of the hexagon in problem 15 when $r = 10$.

21. Find the perimeter of the decagon in problem 16 when $r = 10$.

22. In each of the four preceding problems find the difference between the perimeter of the polygon and the circumference of the circle. How does this difference change as the number of sides of the polygon increases? Do you think this difference can be made less than .1? Less than .001? Less than .000001?

23. In a circle of a given radius, does the perimeter of the regular inscribed polygon increase or decrease as the number of sides increases? What do you think will be true of this perimeter as the number of sides becomes very large, say 1,000,000?

CHAPTER X

QUADRATIC EQUATIONS

67. A problem involving a quadratic equation. In making books and other rectangular objects there seems to be a certain relation between the length and the width of the rectangle which makes it most pleasing to the eye. An artist generally determines the relative length and width by trial, but it is thought that the exact relation may be found by dividing a line into two parts so that the ratio of the whole line to the larger part equals the ratio of the larger part to the smaller part. If, then, the rectangle is made with its length the larger segment of the line and its width the smaller segment, the rectangle is thought to be in most pleasing proportion.

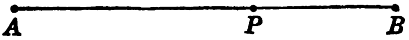

FIG. 23.

If AB, Figure 23, is half the perimeter of the desired rectangle, it is necessary to locate point P so that $\frac{AB}{AP} = \frac{AP}{PB}$; or, if AP is x and AB is 2 ft. so that $\frac{2}{x} = \frac{x}{2-x}$. To find the length of AP it is necessary to solve this equation for x.

Clearing of fractions, $4 - 2x = x^2$.
Transposing, $x^2 + 2x - 4 = 0$.

Since this equation contains the second power of the unknown and no higher power it is called an equation of the second degree, or more commonly a **quadratic equation.** In order that the pupil may learn to solve this problem and many other applied problems he must learn how to solve a quadratic equation.

Dividing a line into two segments such that the whole line is to the greater segment as the greater segment is to the smaller segment is said to be dividing the line into **extreme and mean ratio,** or finding the **golden section** of the line. Later the pupil will find this useful in inscribing a regular decagon in a circle, but further study of geometry is necessary before that problem can be solved.

68. To solve a quadratic equation. Find two values of x which satisfy the equation $x^2 = 9$. What was done to each member of the equation to get the equation $x = \pm 3$? We have here made use of the following

Axiom. Positive square roots of equal numbers are equal.

Solve for $(x - 2)$ the equation $(x - 2)^2 = 25$.

Solve the resulting equations for x.

Solve for $(x + 1)$ the equation $x^2 + 2x + 1 = 9$. You should get two values for $x + 1$. Solve each of these resulting equations for x.

Solve for a, $(a - 3)^2 = 4$.
Solve for a, $a^2 - 6a + 9 = 4$.

Solve, $a^2 + 6a = 40$. Put this equation into the same form as the preceding by adding the same number to each member.

How did you decide what number to add to each member in order to make the left member a square? What added to $m^2 + 14m$ gives the square of a binomial? To $x^2 - 4x$? To $y^2 + 10y$? To $a^2 - 3a$? To $x^2 + x$? To $4x^2 + 4x$? To $9b^2 - 12b$?

Make a rule for finding the term to be added to any two terms of the form $ax^2 + bx$ to complete the square of a binomial.

Solve for x, $x^2 - 4x = 5$.

Solve for n, $2n^2 + 7n = 15$. Hint. First multiply both members by 2. Why?

Example 1. Solve $3y^2 + 10y - 8 = 0$.

Solution.

Transpose. $3y^2 + 10y = 8$.
Multiply by 3. $9y^2 + 30y = 24$.
Complete the square by adding 25 to each member.
$9y^2 + 30y + 25 = 49$.

Taking the square root of each member, $3y + 5 = +7$, or $3y + 5 = -7$.

Solving for y, $y = \frac{2}{3}$, or $y = -4$.
Check. $3\cdot(\frac{2}{3})^2 + 10\cdot\frac{2}{3} = \frac{4}{3} + \frac{20}{3}$
$= 8$.
$3\cdot(-4)^2 + 10(-4) = 48 - 40$
$= 8$.

Rule. *When given the first two terms of the square of a binomial, to find the third term divide the second term by twice the square root of the first term and square the quotient.*

Example 2. Solve $2n^2 - 8n + 3 = 0$.

Solution.

Transpose. $2n^2 - 8n = -3$.
Multiply by 2. $4n^2 - 16n = -6$.
Complete the square. $4n^2 - 16n + 16 = 10$.
Take the square root.
$2n - 4 = +\sqrt{10}$ or $-\sqrt{10}$.
Transpose. $2n = 4 + \sqrt{10}$ or $4 - \sqrt{10}$.
Divide by 2. $n = \frac{4 + \sqrt{10}}{2}$ or $\frac{4 - \sqrt{10}}{2}$.

Exercise 73

Solve the equations:

1. $y^2 = 8$.
2. $a^2 - 10a + 25 = 49$.
3. $n^2 + 6n = 7$.
4. $x^2 - 12x = -11$.
5. $a^2 + 14a + 33 = 0$.
6. $x^2 - 8x = 9$.
7. $4x^2 + 4x = 3$.
8. $9a^2 - 6a = 35$.
9. $4r^2 - 6r - 4 = 0$.
10. $y^2 = 12 - y$.
11. $c^2 - c = 90$.
12. $4x^2 = 7x + 15$.
13. $4a^2 + a - 5 = 0$.
14. $2s^2 - 3s = 2$.
15. $3b^2 + 8b - 3 = 0$.
16. $1 - n^2 - 3n = 0$.
17. $\frac{3}{(x - 1)} = \frac{(x - 1)}{3}$.
18. $c + \frac{1}{c} = 2 + \frac{1}{2}$.
19. $m - 1 + \frac{1}{m-1} = 3$.

20. $a^2 + .3a - .1 = 0.$

21. Solve for x, $x^2 = a^2$.

22. Solve for n, $n^2 - 2bn = -b^2 + c^2$.

23. Solve for y, $6y^2 - 7ry + 2r^2 = 0$.

24. Solve for x, $ax^2 + 2x = 2 + a$.

25. Solve for x, $ax^2 + bx = 0$.

69. The quadratic formula. In order to be a quadratic equation, an equation must contain one term having the second power of the unknown number. This term may have any coefficient not zero. Using x as the unknown write a term having the square of the unknown and a general coefficient. What other terms may the quadratic equation have? The quadratic equation may then always be put into the form $ax^2 + bx + c = 0$. This is called the general quadratic equation. If this equation is solved for x in terms of the coefficients a, b, and c, the result may be used as a formula for solving any other quadratic equation. See if you can solve the equation $ax^2 + bx + c = 0$ before you read the solution which follows.

Solution of the general quadratic equation.

	(1)	$ax^2 + bx + c = 0.$
Multiply by $4a$.	(2)	$4a^2x^2 + 4abx + 4ac = 0.$
Transpose.	(3)	$4a^2x^2 + 4abx = -4ac.$
Complete the square.	(4)	$4a^2x^2 + 4abx + b^2 = b^2 - 4ac.$
Take the square root.	(5)	$2ax + b = +\sqrt{b^2 - 4ac}$ or $-\sqrt{b^2 - 4ac}.$
Transpose.	(6)	$2ax = -b + \sqrt{b^2 - 4ac}$ or $-b - \sqrt{b^2 - 4ac}$
Divide by $2a$.	(7)	$x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

These answers are usually together, thus, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The sign ± is read "plus or minus."

Example. Solve the equation $2x^2 = 3x + 5$. Write the equation in the general form, $2x^2 - 3x - 5 = 0$. In the formula obtained as the value of x in the general quadratic, we here have $a = 2$, $b = -3$, and $c = -5$.

$$\text{Then } x = \frac{-b \pm\sqrt{b^2-4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4\cdot 2\cdot(-5)}}{2\cdot 2}$$

$$= \frac{3 \pm \sqrt{49}}{4} = \tfrac{5}{2}, \text{ or } -1.$$

With practice this solution can be much shortened. See what steps you can omit.

Exercise 74

Solve, using the formula:

1. $x^2 + 3x + 1 = 0$.
2. $2n^2 + 5n + 1 = 0$.
3. $3s^2 + 4s = 5$.
4. $6a^2 - a - 5 = 0$.
5. $7n^2 = 3n + 4$.
6. $7r = 3r^2 - 15$.
7. $3x^2 - 6x = 0$.
8. $3c^2 - c - 2 = 0$.
9. $3n^2 - 12 = 0$.
10. Solve for x, $5x^2 + 2mx - 16m^2 = 0$.
11. Solve for n, $4an - 10a^2n^2 = -3$.
12. $.2c^2 - .44c - .15 = 0$.

Exercise 75

1. Find the two parts of 18 whose product is 72.

2. The length of a certain rectangle is 5 feet more than its width and its area is 104 square feet. Find its length and width.

3. A certain triangle has its base three times its altitude. Its area is $16\frac{1}{2}$ sq. ft. more than the area of a similar triangle whose altitude is 1 ft. shorter than that of the first triangle. Find the base and altitude of each.

4. One leg of a right triangle is 2 ft. longer than the other. The hypotenuse is 10 ft. Find the length of each leg.

5. The longer leg of a right triangle is 20 ft. and the hypotenuse is 5 ft. less than double the shorter leg. Find the hypotenuse.

6. Construct a rectangle 6 in. by 8 in. Within it lay off a margin of uniform width such that the area of the

margin shall be one-half the area of the whole rectangle. To do this it is necessary to find the width of the margin.

7. The side of a square is a in. How wide is the strip of uniform width in the margin of the square if its area is half the area of the square? Use this result as a formula to find the width of such a margin in a square whose side is 8 ft.

8. Three boys agree to mow a square lawn 60 ft. square. How wide a strip around the square shall the first boy mow to mow $\frac{1}{3}$ of the square?

9. The radius of a circle is 6 in. By how much must the radius be reduced to get a circle of half its area?

10. The number of square feet in the area of a certain circle is equal to the number of feet in the circle. How long is its radius?

11. The number of square feet in the area of a certain circle is 8 more than the number of feet in the circle. What is its radius?

70. Special cases of quadratic equations. If $c = 0$ in the equation $ax^2 + bx + c = 0$, the equation becomes $ax^2 + bx = 0$. This equation may at once be solved by factoring. Factoring the left member, we have

$$x(ax + b) = 0.$$

Setting each factor equal to zero,

$$x = 0;$$

or $$ax + b = 0.$$

Transposing in the last equation,

$$ax = -b;$$

and dividing by a, $$x = -\frac{b}{a}.$$

The two roots of the equation are then 0 and $-\frac{b}{a}$.

If $b = 0$ in the equation $ax^2 + bx + c = 0$, the equation becomes $ax^2 + c = 0$.

Transposing c, $ax^2 = -c.$

Dividing by a, $x^2 = -\frac{c}{a}.$

Taking the square root,

$$x = \pm\sqrt{-\frac{c}{a}}.$$

EXAMPLE 1. Solve for x, $5x^2 - 80 = 0.$

Transposing, $5x^2 = 80.$

Dividing by 5, $x^2 = 16.$

Extracting the square root,

$$x = \pm 4.$$

Let the pupil check these roots.

EXAMPLE 2. Solve for x, $6x^2 + 5x = 0.$

Factoring, $x(6x + 5) = 0.$

Setting the factors equal to zero, $x = 0$, or $6x + 5 = 0.$

From the last equation, $6x = -5,$

and $x = -\frac{5}{6}.$

Let the pupil check these results.

Exercise 76

Solve and check the results:

1. $x^2 = 25.$
2. $2x^2 - 3x = 0.$
3. $3x^2 - 1 = 0.$
4. $6x^2 - 5 = 0.$
5. $6x^2 = 0.$
6. $x + x^2 = 0.$
7. $a^2 = a.$
8. $0.05x^2 - 3x = 0.$
9. $0.25m^2 - 1 = 0.$
10. $\frac{2}{3}n^2 - 5n = 0.$
11. Solve $x^2 = 25b^2$ for x.
12. Solve $mx^2 - n = 0$ for x.
13. Solve $a^2y^2 - 1 = 0$ for y.
14. Solve $kz^2 = 0$ for z.
15. Solve $x^2 - .25a^2 = 0$ for x.
16. Solve $ax^2 - bx = 0$ for x.
17. Solve $\frac{2}{3}x^2 - \frac{1}{2}x = 0$ for x.
18. Solve $\frac{4}{5}x^2 - 5k^2 = 0$ for x.
19. Solve $x^2 = a^2 + 2ab + b^2$ for x.
20. Solve $y^2 - a^2 + 2a - 1 = 0$ for y.

71. Finding new formulas from given formulas. One of the important uses of algebra is in solving formulas for the different letters involved, thus obtaining new formulas from given ones. Thus, the formula $A = \pi r^2$, gives the area of a circle when the radius is known. But if we wish to find the radius when the area is known we need the formula $r = \sqrt{\frac{A}{\pi}}$. In the following exercises are given certain formulas which are much used in practice.

Exercise 77

Solve for the letters indicated:

1. The formula $I = \frac{1}{2}mv^2$ is used in physics. Solve it for v.

2. The formula $T = \frac{1.5WL^2}{R}$ is used in architecture. Solve it for L.

3. The formula $L = \dfrac{925}{1 + \dfrac{l^2}{1100d^2}}$ is used to determine the safe load that can be placed upon oak pillars. Solve the equation for d; solve it for l.

4. If a ball is thrown upward with a velocity v' feet per second, the distance of the ball from the earth, measured in feet, after t seconds, is given by the formula

$$d = v't - \tfrac{1}{2}gt^2,$$

where $g = 32.2$. Solve the equation for t. If a ball is thrown upward with a velocity of 60 ft. per second, after how many seconds will it be 40 ft. from the ground?

5. How long will it take the ball described in problem 4 to reach the ground? Hint: Let d equal zero and solve for t.

6. If a ball is thrown downward with a velocity v' the distance it passes over in t seconds is given by the formula

$$d = v't + \tfrac{1}{2}gt^2.$$

Solve this equation for t. If a ball is thrown downward with a velocity of 30 ft. per second, how long will it take it to descend 300 ft.?

7. If a ball is dropped from a balloon the distance it falls in t seconds is given by the formula $d = \frac{1}{2}gt^2$. How long will it take for a bomb dropped from an airplane to fall 1000 ft.?

8. The formula $R = .0042V^2A$ gives approximately the air resistance, R, in pounds, which must be overcome by an automobile traveling V miles per hour, if A is the area in square feet of the front of the automobile. An automobile is traveling 20 miles per hour. Find the air resistance if $A = 26$ sq. ft. What will be the effect upon the air resistance of doubling the speed?

9. Solve the equation $R = .0042V^2A$ for V.

10. The velocity with which water flows through an orifice, as a hole in the side of a water tank, is given by the formula $V = \sqrt{2gh}$, where V is the velocity in feet per second, $g = 32.2$, and h is the height in feet of the level of the water above the center of the orifice. Find V when $h = 10$; when $h = 20$. Increasing h will have what effect upon V? Does doubling h also double V?

11. The formula $Q = .61A\sqrt{2gh}$ gives the number of cubic feet of water, Q, that will flow in one second through an orifice of area A sq. ft., h and g having the same meanings as in the previous problem. Find the number of cubic feet of water flowing through a circular orifice of radius 6 in., whose center is 12 feet below the surface of the water.

12. In the formula $Q = .61A\sqrt{2gh}$, what will be the effect upon Q of increasing A? Of increasing h? Will doubling A double Q?

13. The working load of a steel-wire rope, that is, the load that the rope can be used to lift constantly, is computed from the formula $L = .450C^2$, where L = the working load in tons, and C = the circumference of the rope in inches. Find the working load of a rope 1 in. in diameter.

What will be the effect upon L of doubling the diameter of the rope?

14. From the formula $L = .450C^2$, derive a formula for finding the diameter, D, of a steel-wire rope that can carry a working load of L tons. Find, to the nearest half inch, the diameter of a steel-wire rope that can carry a working load of 10 tons.

Exercise 78. Review

1. Find the sums of (a) $3x$, $-5x$, $\frac{2}{3}x$; (b) $3(m - n)$, $-(m - n)$, $\frac{1}{2}(m - n)$; (c) $6 \cdot 5$, $-2 \cdot 5$, 5; (d) ab, $2b$, $-nb$; (e) xy, $-y$, y.

2. In each of the following pairs of numbers tell which is the greater, and which has the greater absolute value: (a) 4, −7; (b) −8, −1; (c) 0, −2; (d) −4, 4.

Complete the following indicated multiplications:

3. $6(2a^2 - \frac{5}{6}ab + \frac{3}{2}b^2)$.

4. $5(2 \cdot 5^2 - 3 \cdot 5 + 1)$.

5. $(x + \frac{2}{3})\ (x - \frac{1}{2})$.

6. $3ab \cdot \dfrac{3}{ab}$.

7. $(-2)^2 \cdot (-1)^3 \cdot (-4)$.

8. $(x - y)\ (x^2 + xy + y^2)$.

9. Simplify: $a - (a - b) + [-2a + (3a + b) - (-a + b)]$.

Complete the following indicated divisions:

10. $(m^3 + n^3) \div (m + n)$.

11. $(a - b)^3 \div (a - b)$.

12. $\dfrac{8x^2 + 3x - 2}{-2}$.

13. $12a^4b^5c^5 \div (2ab^2c^3)^2$.

14. $(7 \cdot 3^3 - 5 \cdot 3^2 + 3) \div (-3)$.

15. $(10a^2 + 13ab - 3b^2) \div (5a - b)$.

16. Show that $\dfrac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \dfrac{x^5}{1 - x}$.

How many of these terms must be taken to have the quotient correct to .001 when $x = .5$? When $x = .1$? When $x = .01$?

Simplify the following:

17. $(m + n) \div \dfrac{1}{m + n}$.

18. $ab\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$.

19. $\dfrac{n - 1}{n^2 - 4} + \dfrac{n^2 - 4n + 3}{n^2 - n - 6}$.

20. $1 + a + a^2 + \dfrac{a^3}{1 - a}$.

21. $\dfrac{1}{x + \frac{1}{3}}$.

22. $\dfrac{1 + a}{1 + \dfrac{1}{a}}$.

23. What is the H.C.F. of $a^2b^2c^2$, b^3c^3, and a^3c^3? What is the L.C.M.?

24. Find the H.C.F. and the L.C.M. of $p + prt$, p^3, and $p^2 - p^2r^2t^2$.

25. Solve by substitution the equations $3y = x - 7$ and $3x = 5y + 13$.

26. Solve the equations $\dfrac{x}{2} - \dfrac{y}{3} = 1$ and $\dfrac{x}{7} + \dfrac{8y}{21} = 2$.

27. What are the factors of the difference of the squares of x^2 and a?

28. Write two binomials that have a common term n. Find the product of these binomials.

29. Fill the parentheses in the expression $16y^2 - 48y + (\ )$ so that the resulting expression is the square of a binomial. Find its square root.

30. Solve the equation $x^2 + px + q = 0$ for x.

31. Use the result of the previous problem as a formula for solving these quadratic equations:

Exercise 79. Review

1. Write a general linear equation in one unknown and solve it. Use the result as a formula for solving the equation $3x + 5 = 0$.

2. How is an equation cleared of fractions?

3. Plot the graphs of the two equations $x = -4$ and $y = 2$. Find the common solution of the two equations from the graphs.

4. Give an example of a linear equation in two unknowns. Define a solution of such an equation. Find a solution of the example you have given. How many solutions can be found for the equation $x - 2y = 4$? What is the graph of this equation? How many points must be found to determine this graph?

5. Draw a pair of coördinate axes XX' and YY', which intersect at O and divide the paper into four parts. Show the part in which a point is located if (a) its coördinates are both positive; (b) both negative; (c) the abscissa is positive and the ordinate negative; (d) the ordinate is positive and abscissa negative; (e) both zero; (f) the abscissa zero; (g) the ordinate zero.

6. Plot the graphs of the following pairs of equations:

(a) $x + y = 4$, (b) $x = 4$, (c) $x = 2$, (d) $x = 3$,
$x - y = 2$. $y = 6$. $x = 3$. $2x = 6$.

In which cases do the graphs of the pair intersect?

7. Solve for x the equation $3x^2 - 4x - 9 = 0$. Find the values of x correct to .01.

8. One of the values of t found by solving the equation $8000 = 2000t - 16t^2$ for t gives the number of seconds needed for a ball to rise to a height of 8000 feet if shot vertically upward with a velocity of 2000 ft. per second. Solve this equation for t. You should find two values of t. Do you know what they both mean?

9. The difference between a certain number and its reciprocal is $\frac{5}{6}$. Find the number.

Exercise 80. Review

1. Two automobiles that are M miles apart start towards each other, the first running at the average rate of r miles per hour, and the second at the average rate of r' miles per hour.

(a) Write a formula for finding the number of hours H before they start together and meet.

(b) Write a formula for finding the distance D they are apart at the end of H hours.

(c) After they have been running t hours, the first has an accident and is delayed t' hours. Write a formula for finding the number of hours, T, required for them to meet after the first has resumed the journey.

2. In a bicycle race around a circular half-mile track one contestant rides at the rate of 1760 ft. per minute and another at the rate of 1584 ft. per minute. If they continue at this rate how many minutes before the faster will overtake the slower? Write a formula for finding the number of minutes, t, for the faster to overtake the slower if the faster rides at the rate of f feet per minute and the slower at the rate of f' feet per minute.

3. Copper wire costs K cents per pound. A cubic inch of copper weighs p pounds. Make a formula for finding the cost, C, in dollars, of the copper wire in a telephone cable l feet long, containing n wires each of diameter d inches.

4. Reduce the formula

$$P'' = \frac{P'}{2}\left[\frac{1 + \frac{v}{v'}}{1 + \frac{v}{2v'}}\right]$$

to a more convenient form for computation.

5. Solve the equation $H = 1082 + .305t$ for t.

6. Solve $r = \frac{m + i}{1 + i}$ for i.

7. The formula $A = \sqrt{s(s-a)\ (s-b)\ (s-c)}$ gives the area, A, of a triangle whose sides are a, b, and c. In this formula s equals one-half of the sum of the sides. Find the area of a triangle whose sides are 10 in., 15 in., and 17 in., correct to .01 sq. in.

8. In the equation $a = \frac{1}{4}r\sqrt{10 + 2\sqrt{5}}$ find the ratio of a to r correct to .01.

9. If l = length of a water pipe in feet,
d = the inner diameter in inches,
H = height of water above center of pipe,
V = velocity of water in feet per second,

then $$V = \sqrt{\frac{(H + 2500)d}{13.9l}}.$$

This formula is used to find the velocity of water flowing through a pipe which leads from a standpipe or other reservoir.

(*a*) Find the velocity of water flowing through a pipe 6 in. in diameter, 1200 ft. long, when H is 60 ft.

(*b*) Does the velocity increase or decrease with the length of the pipe?

(*c*) Will the velocity be increased if the diameter of the pipe is increased?

(*d*) What will be the effect upon V of increasing H?

(*e*) Can you give reasons for answers to (*b*), (*c*) and (*d*)?

10. This rule is used in practice for finding the length of the belting connecting two pulleys: Add the diameters of the pulleys in inches and divide by 2. Multiply this quotient by 3.25. Add this product to twice the distance in inches between the centers of the pulleys. Divide this result by 12. This quotient is the length of the belting on the pulleys.

(*a*) State this rule as a formula, using D and d as diameters, and l as distance between centers, all expressed in inches.

(*b*) Find the length of the belting connecting two pulleys 12 in. and 18 in. in diameter, whose centers are 12 ft. apart.

(*c*) Find the length of the belting connecting two pulleys 16 in. and 22 in. in diameter whose centers are 16 ft. 8 in. apart.

Exercise 81

1. The line AB, Figure 24, is 36 inches in length.

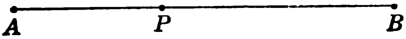

FIG. 24.

A point P divides the line AB into two segments AP and PB. Let the ratio of AP to PB be denoted by r. Then $r = \frac{\text{AP}}{\text{PB}}$. Find r when AP = 6 in.; when AP = 8 in.; 12 in.; 18 in.; 30 in.; 35 in. As P moves along AB from A to B, does AP increase or decrease? Answer the same question for PB? Where is P when $r = 1$? When $r = 0$? What is true of the size of r when P is very near B?

2. In Problem 1 let AB = 36 in. Let the point P move from A to B. Let the distance AP $= x$. Then PB $= 36 - x$, and $r = \frac{x}{36 - x}$. Make a table showing the values of r for integral values of x from 0 to 35. On coördinate paper plot the curve $r = \frac{x}{36 - x}$, using x for the abscissa and r for the ordinate. How does r change as x changes from 0 to 35? How much does r change as x changes from 0 to 1? From 1 to 2? From 10 to 11? From 34 to 35? Where is r changing most rapidly? Find the value of r when x has each of these values: 35.5, 35.9, 35.99, 35.999. Does r have a value when $x = 36$?

3. The length of the side of a square is a. Because of an error in measuring, the side is taken to be $a + x$. What error did that make in the area? The side of a square is

10 in. In computing the area the side is taken as 10 ft. 1 in. What error does that make in the area? Use the result found in the first part of the problem to compute this error.

4. The sides of a rectangle are a and b. In finding the area the sides are taken to be $a + x$ and $b + y$. What error does that make in the area? Illustrate by a figure. What does the error become when $y = 0$? Use the result you have found to find the error when the sides of a field 20 rods by 40 rods are taken to be 20 rods 8 feet, and 40 rods 12 feet.

5. The horse power of certain types of gasoline engines is computed by the formula $\text{HP} = \frac{nd^2}{2.5}$, where n is the number of cylinders and d is the diameter of the cylinders. Does the horse power increase as n increases? As d increases? Will doubling the number of cylinders double the horse power? Will doubling the diameter of the cylinders double the horse power? Which has the greater horse power, a 6-cylinder Packard automobile with $4\frac{1}{2}$ in. cylinders, or a 12-cylinder Packard with 3 in. cylinders?

6. The safe load for rectangular wooden pillars of white oak is given by the formula,

$$\text{T} = \frac{925}{1 + \frac{l^2}{1100d^2}},$$

where T is the safe load in tons, l is the length of the pillar in inches, and d is the width of the smallest side in inches. Find the safe load that can be placed upon a white oak pillar 16 ft. long, and 10 in. square. How will T vary as l increases? As d increases?

7. If a weight of w pounds attached to a string r feet long is swung in a circle with a velocity of v feet per second, then the outward pull (centrifugal force) F, in pounds, is given by the formula,

$$\text{F} = \frac{wv^2}{32r}.$$

Find the centrifugal force when a weight of 2 pounds is swung around a circle of radius 6 feet, with a velocity of 30 ft. a second. What is the effect upon the centrifugal force of an increase in the weight? Of an increase in the velocity? Of an increase in the length of the string? What is the effect upon the centrifugal force of doubling the weight? Of doubling the velocity? Of doubling the length of the string?

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